SKOLIAD No. 104

Robert Bilinski

Please send your solutions to the problems in this edition by ***March 1, 2008***. A copy of **MATHEMATICAL MAYHEM Vol. 6** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Le concours de ce mois-ci proviennent du 23[ième] Concours W.J. Blundon de Mathématiques. Je remercie Don Rideout de l'Université Mémorial de Terre-Neuve qui a eu l'amabilité de me fournir les questions.

23[ième] Concours W.J. Blundon de Mathématiques

parrainé par la Société Mathématique du Canada et le département de mathématique et de statistique de l'Université Mémorial de Terre-Neuve

22 Février 2006

1. Si $\log_a x = \log_b y$, montrer que chacun d'eux égale $\log_{ab} xy$.

2. De combien de manières peut-on changer un billet de $20$$ en 25 sous et 10 sous, si on utilise au moins un de chaque sou ?

3. Si une femme quitte lors d'une fête, alors 20% des gens restant sont des femmes. Si, à la place, une femme se joint à la fête, alors 25% des gens restant sont des femmes. Combien d'hommes se trouvent à la fête ?

4. Trouver deux facteurs de $2^{48} - 1$ qui se trouvent entre 60 et 70.

5. Les changements annuels de population dans une ville dans quatre années consécutives sont respectivement 25% de plus, 25% de plus, 25% de moins et 25% de moins. Trouvez le pourcentage de changement net sur les quatre ans au pourcent le plus près.

6. Si $x + y = 5$ et $xy = 1$, trouver $x^3 + y^3$.

7. Le point $(4, 1)$ est sur la droite passant par $(4, 1)$ et perpendiculaire à la droite $y = 2x + 1$. Trouvez l'aire du triangle formé par la droite $y = 2x + 1$, sa perpendiculaire et l'axe des x.

8. Un point est choisi au hasard à l'intérieur d'un triangle équilatéral. À partir de ce point, on trace les trois perpendiculaires aux côtés. Montrez que la somme de ces trois segments est de même longueur qu'une hauteur du triangle.

9. Trouver tous les triplets d'entiers positifs (x, y, z) satifaisant aux équations

$$x^2 + y - z = 100 \quad \text{et} \quad x + y^2 - z = 124 \,.$$

10. Combien de racines a l'équation $\sin x = \frac{1}{100}\, x$?

23rd W.J. Blundon Mathematics Contest
Sponsored by the Canadian Mathematical Society and lThe Department of Mathematics and Statistics Memorial University of Newfoundland
February 22, 2006

1. If $\log_a x = \log_b y$, show that each is also equal to $\log_{ab} xy$.

2. In how many ways can 20 dollars be changed into dimes and quarters, with at least one of each coin used?

3. If one of the women at a party leaves, then 20% of the people remaining at the party are women. If, instead, another woman arrives at the party, then 25% of the people at the party are women. How many men are at the party?

4. Find two factors of $2^{48} - 1$ between 60 and 70.

5. The yearly changes in the population census of a town for four consecutive years are, respectively, 25% increase, 25% increase, 25% decrease, and 25% decrease. Find the net percent change to the nearest percent over the four years.

6. If $x + y = 5$ and $xy = 1$, find $x^3 + y^3$.

7. The point $(4, 1)$ is on the line that passes through the point $(4, 1)$ and is perpendicular to the line $y = 2x + 1$. Find the area of the triangle formed by the line $y = 2x + 1$, the given perpendicular line, and the x-axis.

8. An arbitrary point is selected inside an equilateral triangle. From this point perpendiculars are dropped to each side of the triangle. Show that the sum of the lengths of these perpendiculars is equal to the length of the altitude of the triangle.

9. Find all positive integer triples (x, y, z) satisfying the equations

$$x^2 + y - z = 100 \quad \text{and} \quad x + y^2 - z = 124 \,.$$

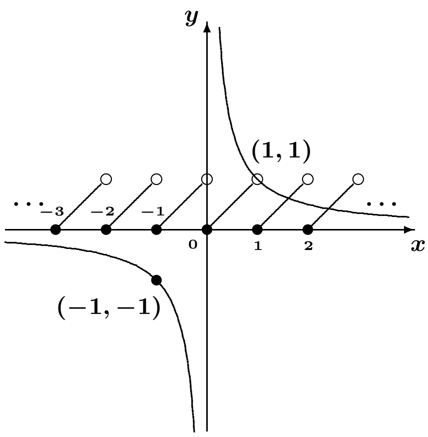
10. How many roots are there to the equation $\sin x = \frac{1}{100}\, x$?

Next we give the solutions to the Concours Montmorency 2004–2005 run by Collège Montmorency [2007 : 3–5].

1. The golden ratio $N = \frac{1+\sqrt{5}}{2} \approx 1.618033989\ldots$ has the remarkable property that its multiplicative inverse $1/N$ is equal to its decimal part $0.618033989\ldots$. Find another number with this property.

Two interesting solutions were presented with a nice generalization. We first present the solution by Daniel Tsai, student, Taipei American School, Taipei, Taiwan.

A non-zero real number x has this remarkable property if and only if $x - \lfloor x \rfloor = 1/x$. Thus, the set X of all such non-zero real numbers is the set of x-coordinates of the points of intersection of the graphs of $y = x - \lfloor x \rfloor$ and $y = 1/x$, as illustrated.



Clearly, no such non-zero real numbers are negative. Furthermore, all the positive solutions are given by $\frac{1}{2}\left(k + \sqrt{k^2+4}\right)$ for positive integers k, as can be seen by solving for the intersection of the graphs of $y = x - k$ and $y = 1/x$.

Next we feature the solution by Justin Yang, student, Lord Bing Secondary School, Vancouver, BC.

Let N be a number having the remarkable property in the problem. Hence, $N - \lfloor N \rfloor = 1/N$. When $\lfloor N \rfloor = n$, we have $N - n = 1/N$. Using the quadratic formula, we get

$$N = \frac{n + \sqrt{n^2+4}}{2}.$$

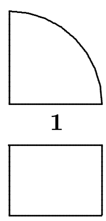
If $n = 1$, we get the given solution $N = \frac{1+\sqrt{5}}{2}$, the golden ratio. If $n = 2$, we get $N = 1 + \sqrt{2}$, a new solution, as required.

Also solved by JONATHAN LOVE, student, Queen Elizabeth Jr.–Sr. High School, Calgary, AB; and VEDULA N. MURTY, Dover, PA, USA. There was one incorrect solution submitted.

2. Consider a quarter circle of radius 1.

(a) Find a rectangle having the same area and the same perimeter as the quarter circle.

(b) For a complete circle of radius 1, is it possible to find such a rectangle, having an area and a perimeter equal to that of the circle? Justify your answer.



Solution to part (a) *by Natalia Desy, student, Palembang, Indonesia.*

Let the rectangle have sides of length x and y. Then the area is $xy = \frac{\pi}{4}$ and the perimeter is $2x + 2y = 2 + \frac{\pi}{2}$. Substituting $y = \pi/(4x)$ into the second equation, we get $4x^2 - (4 + \pi)x + \pi = 0$. Using the quadratic formula, we get $x = \dfrac{4 + \pi \pm (4 - \pi)}{8}$. We may assign either value to x; the other value will be y. Thus, our rectangle has sides 1 and $\frac{\pi}{4}$.

Solution to part (b) *by Vedula N. Murty, Dover, PA, USA.*

It is not possible to find such a rectangle. If the rectangle has to have the same perimeter and area as a circle with radius 1, we have $x + y = \pi$ and $xy = \pi$. Since $(x + y)^2 \geq 4xy$ for all real numbers x and y, we then have $\pi^2 \geq 4\pi$, implying that $\pi \geq 4$, which is false. Therefore, no such real numbers x and y exist.

Also solved by JONATHAN LOVE, student, Queen Elizabeth Jr.–Sr. High School, Calgary, AB; MARIYA SARDARLI, student, McKernan Elementary and Junior High School, Edmonton, AB; and JUSTIN YANG, student, Lord Bing Secondary School, Vancouver, BC. Desy and Murty solved both parts (a) *and* (b).

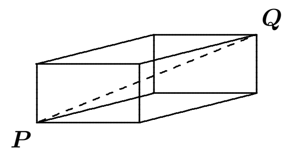
3. A barrel is filled with water. We empty half of its contents and then add a litre of water. After doing this operation seven consecutive times, we are left with three litres of water in the barrel. How many litres were in the barrel at the beginning?

Solution by Natalia Desy, student, Palembang, Indonesia.

Let x be the volume of the barrel, in litres. If we empty half, we are left with $\frac{1}{2}x$, and adding a litre gives $\frac{1}{2}x + 1$. After repeating these operations 7 times, the barrel contains $\frac{1}{128}x + \frac{127}{64}$ liters of water. Setting this equal to 3 and solving, we get $x = 130$. Hence, the initial amount of water in the barrel is 130 litres.

Also solved by JONATHAN LOVE, student, Queen Elizabeth Jr.–Sr. High School, Calgary, AB; and MARIYA SARDARLI, student, McKernan Elementary and Junior High School, Edmonton, AB.

4. The areas of three faces of a rectangular parallelepiped are 18 cm^2, 40 cm^2 and 80 cm^2. Find: (a) its volume; (b) the length of its diagonal $\overline{PQ}$.



Solution by Justin Yang, student, Lord Bing Secondary School, Vancouver, BC.

(a) Assume the rectangular parallelepiped has side lengths a, b, and c $(0 < a < b < c)$. Hence, we have $ab = 18$, $bc = 80$, and $ac = 40$. Multiplying the three equations, we obtain $a^2b^2c^2 = 57600$. The volume is $abc = \sqrt{57600} = 240$ cm^3.

(b) We have $a = abc/(bc) = 240/80 = 3$; similarly, we get $b = 6$ and $c = 40/3$. Hence, $PQ = \sqrt{3^2 + 6^2 + (40/3)^2} = \frac{1}{3}\sqrt{2005}$.

Also solved by NATALIA DESY, student, Palembang, Indonesia; JONATHAN LOVE, student, Queen Elizabeth Jr.–Sr. High School, Calgary, AB; VEDULA N. MURTY, Dover, PA, USA; and MARIYA SARDARLI, student, McKernan Elementary and Junior High School, Edmonton, AB.

5. Evaluate $\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$.

Identical solutions by Natalia Desy, student, Palembang, Indonesia; Justin Yang, student, Lord Bing Secondary School, Vancouver, BC; and Vedula N. Murty, Dover, PA, USA.

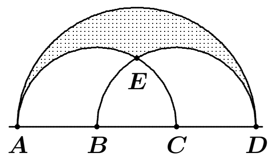
Let $x = \sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$. Then

$$x^2 \;=\; 6+\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}} \;=\; 6+x\,,$$

or $x^2-x-6=0$. Factoring, we get $(x-3)(x+2)=0$, which implies that $x=3$ or $x=-2$. But x is the result of a positive square root; thus, $x=3$.

Also solved by MARIYA SARDARLI, student, McKernan Elementary and Junior High School, Edmonton, AB.

6. Let A, B, C, and D be collinear points such that $\overline{AB} = \overline{BC} = \overline{CD} = 1$. Consider three semi-circles of respective diameters $\overline{AC}$, $\overline{BD}$ and $\overline{AD}$. Let E be the intersection of the semi-circles with centres B and C. Determine the area of the curvilinear triangle AED (shaded in the drawing).



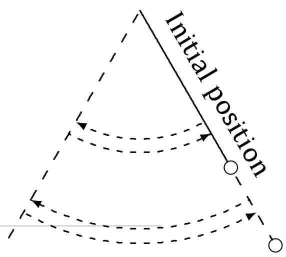
Solution by Justin Yang, student, Lord Bing Secondary School, Vancouver, BC.

Let $[\nabla]$ denote the area of region ∇, and let $\widetilde{XY}$ denote the semi-circle with diameter XY. From the problem statement, we deduce that the large semi-circle has diameter 3 (and radius $\frac{3}{2}$) and that $\triangle BEC$ is equilateral. With this additional information, we can evaluate the required area using the Inclusion-Exclusion Principle:

$$\begin{aligned}
&[\text{curvilinear triangle } AED] \\
&\quad= [\widetilde{AD}] - [\widetilde{AC}] - [\widetilde{BD}] + [\text{curvilinear triangle } BEC] \\
&\quad= \tfrac{1}{2}\left(\tfrac{9}{4}\pi\right) - \tfrac{1}{2}\pi - \tfrac{1}{2}\pi + [\text{curvilinear triangle } BEC] \\
&\quad= \tfrac{1}{8}\pi + [\text{curvilinear triangle } BEC] \\
&\quad= \tfrac{1}{8}\pi + [\text{sector } CBE] + [\text{sector } BCE] - [\triangle BEC] \\
&\quad= \tfrac{1}{8}\pi + \tfrac{1}{6}\pi + \tfrac{1}{6}\pi - \tfrac{1}{4}\sqrt{3} \;=\; \tfrac{11}{24}\pi - \tfrac{1}{4}\sqrt{3}\,.
\end{aligned}$$

Also solved by NATALIA DESY, student, Palembang, Indonesia; JONATHAN LOVE, student, Queen Elizabeth Jr.–Sr. High School, Calgary, AB; and MARIYA SARDARLI, student, McKernan Elementary and Junior High School, Edmonton, AB.

7. The oscillation period of a pendulum is proportional to the square root of its length (for example, to triple the oscillation period, we multiply the length by nine). Two pendulums of different lengths are released from the initial position shown. The shorter one measures 25 cm, and its oscillation period is 1 second. The two pendulums are aligned again for the first time after 7 seconds in their initial position. Find the length of the longer pendulum. (Air resistance is neglected.)



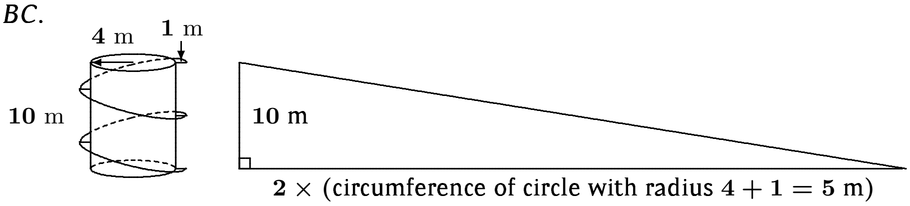
Identical solutions by Natalia Desy, student, Palembang, Indonesia; and Justin Yang, student, Lord Bing Secondary School, Vancouver, BC.

From the shorter pendulum, we have $T = k\sqrt{L}$ or $1 = k\sqrt{25}$, which gives us $k = 1/5$. For the longer pendulum, we get $T = k\sqrt{L}$ or $7 = \frac{1}{5}\sqrt{L}$, which gives us $L = 35^2 = 1225$ cm.

Also solved by MARIYA SARDARLI, student, McKernan Elementary and Junior High School, Edmonton, AB. There was one incorrect solution submitted.

8. In a refinery, a cylindrical storage tank has a spiral staircase one meter wide attached to its exterior. The staircase goes from the bottom to the top while making exactly 2 complete revolutions. If the tank has a height of 10 m and a diameter of 8 m, find the length of the exterior edge of the staircase.

Solution by Justin Yang, student, Lord Bing Secondary School, Vancouver, BC.



The length of the exterior edge of the staircase can be thought of as the hypotenuse of a right-angled triangle obtained by "unrolling" the staircase from around the tank. One leg of the triangle is the height of the tank ($H = 10$ m); the other is twice the circumference of a circle with diameter $D = 8 + 2 = 10$ m. (We take twice the circumference to take into account the two complete revolutions the ramp makes around the tank.)

$$\text{Ramp length} = \sqrt{H^2 + (2\pi D)^2} = \sqrt{100 + 400\pi^2} = 10\sqrt{1 + 4\pi^2}\,.$$

Also solved by JONATHAN LOVE, student, Queen Elizabeth Jr.–Sr. High School, Calgary, AB; and MARIYA SARDARLI, student, McKernan Elementary and Junior High School, Edmonton, AB.

That brings us to the end of another issue. This month's winner of a past volume of Mathematical Mayhem is Justin Yang. Congratulations, Justin! Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier février 2008**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M307. *Proposé par Neven Jurič, Zagreb, Croatie.*

Deux carrés magiques 4×4 ont la propriété que la somme de chacune de leurs lignes, de chacune de leurs colonnes et de leurs deux diagonales donne le même nombre N. On considère alors, pour chaque carré, la somme des éléments de ses quatre coins. Ces sommes peuvent-elles être différentes ou doivent-elles être égales ? (En d'autre termes, la somme des éléments des quatre coins dépend-elle du carré lui-même ou de la *somme magique* N ?) Déterminer cette somme si elle est constante, ou alors montrer que ces sommes peuvent différer.

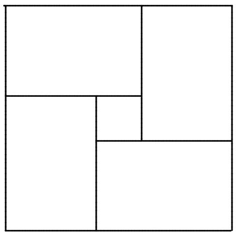
M308. *Proposé par Babis Stergiou, Chalkida, Grèce.*

Soit ABC un triangle rectangle avec $A = 90^\circ$, et soit M le point milieu du côté AB. Si D est le pied de la perpendiculaire de A sur CM et N le point milieu de DC, montrer que $BD \perp AN$.

M309. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Déternimer tous les entiers non négatifs possibles x, y, z et t de sorte que $3^x + 3^y + 3^z + 3^t$ soit un cube parfait.

M310. *Proposé par J. Walter Lynch, Athens, GA, É-U.*



Quatre rectangles congruents sont disposés pour former un carré de telle sorte qu'ils entourent un carré plus petit.

Soit S l'aire du carré extérieur et Q celle du carré intérieur. Si l'aire du carré extérieur est 9 fois celle du carré intérieur, déterminer le rapport des côtés des rectangles.

M311. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a, b et c trois nombres réels positifs, et soit $m \in \left(0, \frac{1}{4}\right)$. Montrer qu'une au moins des équations suivantes possède des solutions réelles :

$$\begin{aligned} ax^2 + bx + cm &= 0\,, \\ bx^2 + cx + am &= 0\,, \\ cx^2 + ax + bm &= 0\,. \end{aligned}$$

M312. *Proposé par G.P. Henderson, Garden Hill, Campbellcroft, ON.*

Jean est en négotiation avec son banquier sur les termes d'une hypothèque. Ils sont tombés d'accord sur le montant L de celle-ci ainsi que sur un taux annuel d'intérêt de i.

Jean propose «Je veux faire des paiements de P dollars à la fin de chaque année pour les prochaines n années. C'est plus qui n'en faut pour payer les intérêts. L'excédent servira à réduire le principal pour l'année suivante. A la fin des n années, je contracterai une nouvelle hypothèque pour le principal restant.»

Le banquier répond «Je préférerais des paiements plus fréquents. Je suggère des paiements de $P/4$ chaque trimestre avec un intérêt de $i/4$ appliqué sur le solde du trimestre précédent.»

Mais Jean s'objecte «Mais alors le taux annuel effectif sera plus grand que i !»

Le banquier rétorque «Oui, mais le montant restant au temps n sera plus petit !»

Jean trouve cela dur à croire. Est-ce vrai ?

M307. *Proposed by Neven Jurič, Zagreb, Croatia.*

Two 4×4 magic squares have the property that all four of their rows, all four of their columns, and their two diagonals all sum to the same value N. Consider the sum of the four corner elements of each square. Can these sums be different, or must they be the same? (In other words, does the corner sum depend on the square itself, or only on the *magic sum* N?) Either determine the constant sum, or show that these sums can differ.

M308. *Proposed by Babis Stergiou, Chalkida, Greece.*

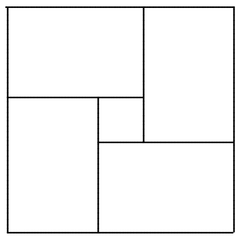
Let ABC be a right triangle with $A = 90^\circ$, and let M be the mid-point of side AB. If D is the foot of the perpendicular from A to CM and N is the mid-point of DC, prove that $BD \perp AN$.

M309. *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all possible non-negative integers x, y, z, and t such that $3^x + 3^y + 3^z + 3^t$ is a perfect cube.

M310. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Four congruent rectangles are arranged in a square pattern so that they enclose a smaller square.



Let S be the area of the outer square and Q the area of the inner square. If the area of the outer square is 9 times the area of the inner square, determine the ratio of the sides of the rectangles.

M311. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers, and let $m \in \left(0, \frac{1}{4}\right)$. Show that at least one of the following equations has real roots:

$$\begin{aligned} ax^2 + bx + cm &= 0\,, \\ bx^2 + cx + am &= 0\,, \\ cx^2 + ax + bm &= 0\,. \end{aligned}$$

M312. *Proposed by G.P. Henderson, Garden Hill, Campbellcroft, ON.*

John is negotiating the terms of a mortgage with his bank manager. They have agreed that the loan will be for L dollars and that the annual interest rate will be i.

John says, "I will make payments of P dollars at the end of each year for the next n years. This is more than enough to pay the interest. The excess will reduce the principal outstanding for the next year. At the end of n years, I will arrange a new mortgage for the remaining principal."

The manager responds, "I would like more frequent payments. I suggest payments of $P/4$ each quarter-year with interest rate $i/4$ applied to the previous quarter's balance."

John objects, "But then the effective annual interest rate will be greater than i!"

The manager replies, "Yes, but the amount outstanding at time n will be less!"

John finds this hard to believe. Is it true?

Mayhem Solutions

M257. *Proposed by Fabio Zucca, Politecnico di Milano, Milano, Italy.*

For a given positive integer k, consider the set of lattice points $\{(x, y)\}$ where x and y are integers such that $0 \leq x \leq 2k+1$ and $0 \leq y \leq 2k+1$. Two points are selected at random from this set. All points have the same probability of being selected and the points need not be distinct. Find the probability that the area of the triangle (possibly degenerate) formed by these two points and the point $(0, 0)$ is an integer (possibly 0).

Solution by Hasan Denker, Istanbul, Turkey.

This problem is a generalization of Mayhem problem M253 in which case k was equal to 3, and can be solved in a similar fashion. Noting that the probability that a randomly selected integer between 0 and $2k+1$ is even (or odd) is $\frac{1}{2}$, and using a similar argument as for M253, we find that the probability that the area of the triangle is an integer is $\frac{5}{8}$. We can therefore conclude that the probability that the area of the triangle is an integer is independent of k.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; D. KIPP JOHNSON, Beaverton, OR, USA; and the proposer. One incorrect solution was also submitted.

M258. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let c, d, and n be integers such that $n = c^2 + d^2$. Prove that $n = (a^2 + b^2)/5$ for some integers a and b.

Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

Take $a = 2c - d$ and $b = 2d + c$. Since c and d are integers, it follows that a and b are also integers. We then have

$$\frac{a^2+b^2}{5} = \frac{(2c-d)^2+(2d+c)^2}{5} = \frac{5(c^2+d^2)}{5} = c^2+d^2 = n\,.$$

Hence, such integers exist by construction.

Also solved by ARKADY ALT, San Jose, CA, USA; HOUDA ANOUN, Bordeaux, France; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JEAN-DAVID HOULE, student, McGill University, Montreal, QC; and D. KIPP JOHNSON, Beaverton, OR, USA.

M259. *Proposed by the Mayhem Staff.*

The number n is formed by concatenating the strings of digits formed by the numbers 2^{2006} and 5^{2006}. How many digits does n have?

Solution by Arkady Alt, San Jose, CA, USA.

More generally, for any natural number m, let p and q be the number of digits in the strings of digits formed by 2^m and 5^m, respectively. Then $10^{p-1} < 2^m < 10^p$ and $10^{q-1} < 5^m < 10^q$. Therefore,

$$\left(10^{p-1}\right)\left(10^{q-1}\right) \;<\; 2^m \cdot 5^m \;<\; 10^p \cdot 10^q\,;$$

that is,

$$10^{p+q-2} \;<\; 10^m \;<\; 10^{p+q}\,.$$

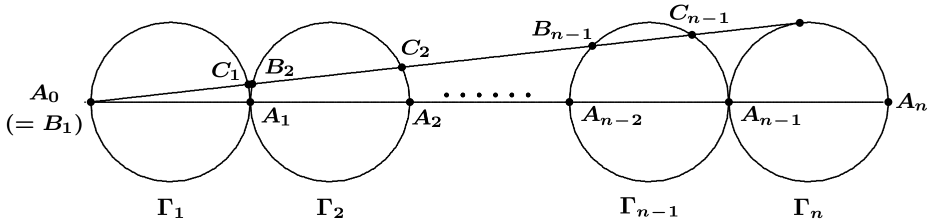
Thus, $p+q-2 < m < p+q$, which is equivalent to $m = p+q-1$, or $p+q = m+1$. We can conclude that a concatenation of 2^m and 5^m has $m+1$ digits. In particular, taking $m = 2006$, we find that n has 2007 digits.

Also solved by HOUDA ANOUN, Bordeaux, France; ALPER CAY, Uzman Private School, Kayseri, Turkey; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JEAN-DAVID HOULE, student, McGill University, Montreal, QC; D. KIPP JOHNSON, Beaverton, OR, USA; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India.

M260. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

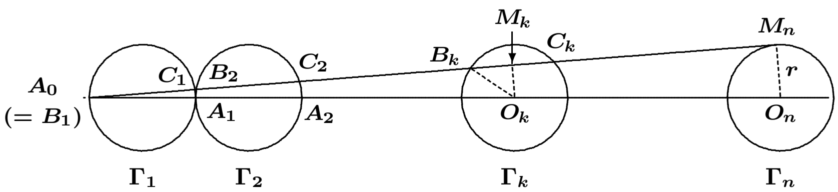
Points A_0, A_1, ..., A_n lie on a line, in that order, spaced a uniform distance $2r$ apart. For $1 \le k \le n$, let Γ_k be the circle with $A_{k-1}A_k$ as diameter. The line through A_0 tangent to Γ_n intersects the circle Γ_k at the points B_k and C_k, for $1 \le k \le n-1$.

Determine the length of the line segment B_kC_k for $1 \le k \le n-1$.



Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Let O_k be the centre of Γ_k. Let M_k be the mid-point of chord B_kC_k for $1 \le k \le n-1$, and let M_n be the point of tangency to Γ_n.



Let k be fixed such that $1 \le k \le n-1$. It can be seen that triangles $A_0O_kM_k$ and $A_0O_nM_n$ are similar. We can conclude that

$$\frac{O_kM_k}{r} = \frac{A_0O_k}{A_0O_n} = \frac{2kr-r}{2nr-r} = \frac{2k-1}{2n-1}\,.$$

If we set $\theta_k = \angle B_kO_kM_k$, then $\cos\theta_k = \frac{O_kM_k}{r} = \frac{2k-1}{2n-1}$. Since $\frac{1}{2}B_kC_k = r\sin\theta_k$, we have

$$\begin{aligned} B_kC_k &= 2r\sin\theta_k = 2r\sqrt{1-\cos^2\theta_k} \\ &= 2r\sqrt{1-\left(\frac{2k-1}{2n-1}\right)^2} = \frac{2r}{2n-1}\sqrt{(2n-1)^2-(2k-1)^2} \\ &= \frac{4r}{2n-1}\sqrt{(n-k)(n+k-1)}\,. \end{aligned}$$

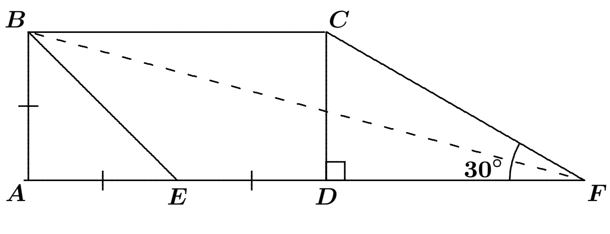
Also solved by KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India. There were two incorrect solutions submitted.

M261. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Rectangle $ABCD$ has $AB = \frac{1}{2}BC$. On the outside of the rectangle, draw $\triangle DCF$, where $\angle DFC = 30^\circ$ and ADF is a straight line segment. Let E be the mid-point of AD.

Determine the measure of $\angle EBF$.

Essentially the same solution by ROBERT BILINSKI, Collège Montmorency, Laval, QC; ALPER CAY, Uzman Private School, Kayseri, Turkey; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India.



In right triangle CDF, we have $\angle DFC = 30^\circ$ and $\angle DCF = 60^\circ$. We can then conclude that $CF = 2CD = 2AB = BC$. Now, considering isosceles triangle BCF, we have $\angle BCF = 150^\circ$ and consequently, $\angle CBF = \angle CFB = 15^\circ$. Also, we know that triangle ABE is isosceles with $\angle ABE = \angle AEB = 45^\circ$. Thus,

$$\angle EBF = 90^\circ - \angle ABE - \angle CBF = 30^\circ\,.$$

Also solved by COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; JEAN-DAVID HOULE, student, McGill University, Montreal, QC; D. KIPP JOHNSON, Beaverton, OR, USA; and GEOFFREY A. KANDALL, Hamden, CT, USA.

M262. *Proposed by Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which $f(1) = 1$ and, for all real numbers x and y, we have $f(x+y) = 3^y f(x) + 2^x f(y)$.

Combination of similar solutions by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; Houda Anoun, Bordeaux, France; Hasan Denker, Istanbul, Turkey; Jean-David Houle, student, McGill University, Montreal, QC; D. Kipp Johnson, Beaverton, OR, USA; and Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Let f be any function satisfying the given conditions $f(1) = 1$ and, for all real numbers x and y,

$$f(x+y) \;=\; 3^y f(x) + 2^x f(y)\,. \tag{1}$$

Setting $y = 1$ in (1) gives, for all $x \in \mathbb{R}$,

$$f(x+1) \;=\; 3f(x) + 2^x f(1) \;=\; 3f(x) + 2^x\,. \tag{2}$$

Setting $x = 1$ in (1) gives, for all $y \in \mathbb{R}$,

$$f(1+y) \;=\; 3^y f(1) + 2f(y) \;=\; 3^y + 2f(y)\,. \tag{3}$$

Changing y to x in (3), we get, for all $x \in \mathbb{R}$,

$$f(1+x) \;=\; 3^x + 2f(x)\,. \tag{4}$$

Finally, using (2) and (4) and noting that $f(x+1) = f(1+x)$, we get

$$3f(x) + 2^x \;=\; 3^x + 2f(x)\,.$$

Thus, $f(x) = 3^x - 2^x$.

Also solved by COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

Problem of the Month

Ian VanderBurgh

Here is a problem that requires only some careful reasoning (albeit pretty tricky careful reasoning) and the ability to add.

Problem (2006 Grade 8 Gauss Contest)

In the diagram, the numbers from 1 to 25 are to be arranged in the 5×5 grid so that each number, except 1 and 2, is the sum of two of its neighbours. (Numbers in the grid are *neighbours* if their squares touch along a side or at a corner. For example, the "1" has 8 neighbours.) Some of the numbers have already been filled in. Which number must replace the "?" when the grid is completed?

			20	21
	6	5	4	
23	7	1	3	?
	9	8	2	
25	24			22

This is not another Sudoku—honest! It looks a bit like one, though. That is part of the reason why this problem was included on the Contest—it is nice to have problems that look familiar but, upon closer examination, are a bit different.

Solution: We could just fiddle around by trial and error until we get some numbers that work. But we will walk through the solution in a logical way.

It's tough to know exactly where to start. First, it makes sense to check which numbers are missing. The grid already includes the numbers 1 to 9 and 20 to 25; so those missing are 10 to 19.

Next, we could figure out which numbers in the grid are already the sum of two neighbours. For example, 9 has neighbours 1 and 8 (and $9 = 1 + 8$); 8 has neighbours 1 and 7 (and $8 = 1 + 7$), and so on. Let's italicize every number which is already the sum of two of its neighbours, as well as the entries 1 and 2.

			20	21
	6	*5*	*4*	
23	*7*	*1*	*3*	?
	9	*8*	*2*	
25	24			22

Now what? It's probably time for that tried and true problem-solving technique—panic. After we get that out of our system, we might try looking at some of the numbers that have almost all of their neighbours already filled in. Also, we might as well focus on the part of the grid near the "?".

For example, consider 21. Since 21 already has neighbours 20 and 4, we must write 21 as either $20 + 1$ or $4 + 17$. But the number 1 already appears elsewhere in the grid; thus, the empty space below 21 must be 17.

			20	*21*
	6	*5*	*4*	17
23	*7*	*1*	*3*	?
	9	*8*	*2*	
25	24			22

Looking at 17 as we did with 21, we see that 17 must be $3 + 14$ or $4 + 13$; thus, the "?" must represent either 13 or 14. But we can't say for sure yet which one it is.

How about 22? It cannot be $2 + 20$, as 20 is already accounted for. What two numbers add to 22 and are not yet in the grid? The only possibility is 10 and 12, in some order. But can we tell which of 10 and 12 is placed where? If 10 was above 22, we could not get 10 as the sum of two neighbours, since $2 + 8$ and $3 + 7$ are not possible. If 12 is above 22, then $12 = 10 + 2$ and $10 = 8 + 2$, which can work.

			20	*21*
	6	*5*	*4*	17
23	*7*	*1*	*3*	?
	9	*8*	*2*	*12*
25	24		*10*	*22*

We know that the "?" is either 13 or 14. Could it be 13? Are there two neighbours of "?" that add to 13? No. So the "?" must be 14, which solves the problem.

But wait! We can't stop now! Let's carry on a bit further.

Looking at 25, we see that 25 must be $24 + 1$ (not a possibility) or $9 + 16$. Hence, the number in the space above 25 must be 16. This now allows us to italicize 23, 24, 25, and 16. (Why?)

			20	*21*
	6	*5*	*4*	*17*
23	*7*	*1*	*3*	*14*
16	*9*	*8*	*2*	*12*
25	*24*		*10*	*22*

Try completing the rest of the grid on your own!

THE OLYMPIAD CORNER

No. 264

R.E. Woodrow

We begin this number of the *Corner* with three problems from the 2003 Kürschák Competition in Hungary. Thanks go to Christopher Small, Canadian Team leader to the IMO in Athens, for collecting them.

2003 KÜRSCHÁK COMPETITION

1. Let EF be a diameter of the circle Γ, and let e be the tangent line to Γ at E. Let A and B be any two points of e such that E is an interior point of the segment AB, and $AE \cdot EB$ is a fixed constant. Let AF and BF meet Γ at A' and B', respectively. Prove that all such segments $A'B'$ pass through a common point.

2. We define a k-colouring of a graph G to be a colouring of its vertices using k possible colours such that the end-points of any edge have different colours. We say that G is uniquely k-colourable if G has a k-colouring and any two vertices which have the same colour in one k-colouring have the same colour in every k-colouring. Prove that if a graph G with n vertices ($n \geq 3$) is uniquely 3-colourable, then the number of its edges is at least $2n - 3$.

3. Prove that the following inequality holds for all positive integers n with the exception of finitely many n:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\gcd(i,j) \;>\; 4n^2\,.$$

Next we give the Seniors Level problems from the 21st Hellenic Mathematical Olympiad "Archimedes" given February 7, 2004. Thanks again go to Christopher Small.

HELLENIC MATHEMATICAL COMPETITIONS 2004
Seniors Level

1. Find the greatest possible value of the positive real number M such that, for all x, y, $z \in \mathbb{R}$,

$$x^4 + y^4 + z^4 + xyz(x + y + z) \;\geq\; M(xy + yz + zx)^2\,.$$

2. Prove that there do not exist positive integers x_1, x_2, $\ldots$, x_m, where $m \geq 2$, such that $x_1 < x_2 < \cdots < x_m$ and $\sum\limits_{i=1}^{m} x_i^{-3} = 1$.

3. A circle (O, r) and a point A outside the circle are given. From A we draw a straight line ε, different from the line AO, which intersects the circle at B and Γ, with B between A and Γ. Next we draw the symmetric line of ε with respect to the axis AO, which intersects the circle at E and Δ, with E between A and Δ.

Prove that the diagonals of the quadrilateral $B\Gamma\Delta E$ pass through a fixed point; that is, they always intersect at the same point, independent of the position of the line ε.

4. Let M be a subset of the natural numbers with 2004 elements. If we know that there is no element in M which is equal to the sum of any two other elements of M, determine the minimum value of the greatest element of M.

Next are the 10 problems of the Vietnamese Mathematical Olympiad in 2004. Thanks go to Christopher Small for collecting them.

VIETNAMESE MATHEMATICAL OLYMPIAD 2004

1. Solve the system of equations

$$\begin{aligned} x^3 + x(y-z)^2 &= 2\,, \\ y^3 + y(z-x)^2 &= 30\,, \\ z^3 + z(x-y)^2 &= 16\,. \end{aligned}$$

2. Solve the system of equations

$$\begin{aligned} x^3 + 3xy^2 &= -49\,, \\ x^2 - 8xy + y^2 &= 8y - 17x\,. \end{aligned}$$

3. Let ABC be a triangle in a plane. The internal angle bisector of $\angle ACB$ cuts the side AB at D.

Consider an arbitrary circle Γ_1 passing through C and D so that the lines BC and CA are not its tangents. This circle cuts the lines BC and CA again at M and N, respectively.

(a) Prove that there exists a circle Γ_2 touching the line DM at M and touching the line DN at N.

(b) The circle Γ_2 from part (a) cuts the lines BC and CA again at P and Q, respectively. Prove that the measures of the segments MP and NQ are constant as Γ_1 varies.

4. Given an acute triangle ABC inscribed in a circle Γ in a plane, let H be its orthocentre. On the arc BC of Γ not containing A, take a point P distinct from B and C. Let D be the point such that $\overrightarrow{AD} = \overrightarrow{PC}$. Let K be the orthocentre of triangle ACD, and let E and F be the orthogonal projections of K onto the lines BC and AB, respectively. Prove that the line EF passes through the mid-point of HK.

5. Consider the sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ defined by $x_1 = 1$ and

$$x_{n+1} = \frac{(2+\cos 2\alpha)x_n + \cos^2\alpha}{(2-2\cos 2\alpha)x_n + 2 - \cos 2\alpha}$$

for every $n = 1, 2, \ldots$, where α is a real parameter. For each $n = 1, 2, \ldots$, let $y_n = \sum\limits_{k=1}^{n} \frac{1}{2x_k+1}$. Determine all values of α so that the sequence $\{y_n\}_{n=1}^{\infty}$ has a finite limit. Find this limit in these cases.

6. Find the least value and the greatest value of the expression

$$P = \frac{x^4+y^4+z^4}{(x+y+z)^4},$$

where x, y, and z are positive real numbers satisfying the condition

$$(x+y+z)^3 = 32xyz.$$

7. Find all triples of positive integers (x, y, z) satisfying the condition

$$(x+y)(1+xy) = 2^z.$$

8. Let A be the set of the first 16 positive integers. Find the least positive integer k satisfying the following condition: in each subset consisting of k elements of A, there exist two distinct elements a and b such that a^2+b^2 is a prime number.

9. Let n be an integer, $n \geq 2$. Prove that for every integer k such that $2n-3 \leq k \leq n(n-1)/2$, there exist n distinct real numbers $a_1, a_2, \ldots, a_n$ such that among all numbers of the form $a_i + a_j$, $1 \leq i < j \leq n$, there exist exactly k distinct numbers.

10. For every positive integer n, let $S(n)$ be the sum of all digits in the decimal representation of n. If m is a positive integral multiple of 2003, find the least value of $S(m)$.

And to round out your problem pleasures, we give the Selected Camp Problems from the 2004 Taiwanese Mathematical Olympiad. Once again, thanks go to Christopher Small, Canadian Team Leader to the IMO in Athens, for collecting them for our use.

2004 TAIWANESE MATHEMATICAL OLYMPIAD
Selected Camp Problems

1. Let $\mathbb{N}_0$ denote the set of non-negative integers. Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(3m + 2n) = f(m) \cdot f(n)$ for all $m, n \in \mathbb{N}_0$.

2. Find all pairs of positive integers (a, b) satisfying

$$\sqrt{\frac{ab}{2b^2 - a}} = \frac{a + 2b}{4b}.$$

3. Suppose that the points D and E lie on the circumcircle of $\triangle ABC$, ray $\overrightarrow{AD}$ is the interior angle bisector of $\angle BAC$, and ray $\overrightarrow{AE}$ is the exterior angle bisector of $\angle BAC$. Let F be the symmetrical point of A with respect to D, and let G be the symmetrical point of A with respect to E. Prove that, if the circumcircle of $\triangle ADG$ and the circumcircle of $\triangle AEF$ intersect at P, then AP is parallel to BC.

4. Let O and H be the circumcentre and orthocentre of an acute triangle ABC. Suppose that the bisectrix of $\angle BAC$ intersects the circumcircle of $\triangle ABC$ at D, and that the points E and F are symmetrical points of D with respect to BC and O, respectively. If AE and FH intersect at G and if M is the mid-point of BC, prove that GM is perpendicular to AF.

5. A one-to-one function $f : \mathbb{Z} \to \mathbb{R}$ is given (where $\mathbb{Z}$ is the set of integers and $\mathbb{R}$ is the set of real numbers). Also given are n different positive integers $a_1, a_2, \ldots, a_n$. Prove that there exists an integer p such that, among the set of $2n$ integers $p - a_1, p + a_1, p - a_2, p + a_2, \ldots, p - a_n, p + a_n$, there are at least n integers b such that $f(b) \geq f(p)$.

6. The seats at the Christmas Feast for the company "Enough" are arranged in a square consisting of 10 rows with 10 seats in each row. All 100 workers have different salaries. Each of them asks all his neighbours (those workers sitting immediately beside him, in front of him, or behind him—four people at most) how much they earn. A worker feels content with his salary only if he has at most one neighbour who earns more than himself. What is the maximum possible number of workers who are satisfied with their salaries?

Next we give an alternate solution to a problem of the Singapore Mathematical Olympiad given in [2005 : 216]. A solution was published last year [2006 : 386].

4. Find all real-valued functions $f : \mathbb{Q} \longrightarrow \mathbb{R}$ defined on the set of all rational numbers $\mathbb{Q}$ satisfying the conditions

$$f(x+y) = f(x) + f(y) + 2xy,$$

for all x, y in $\mathbb{Q}$ and $f(1) = 2002$. Justify your answers.

Alternate Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $g(x) = f(x) - x^2$ for $x \in \mathbb{Q}$. Then, for all x, $y \in \mathbb{Q}$,

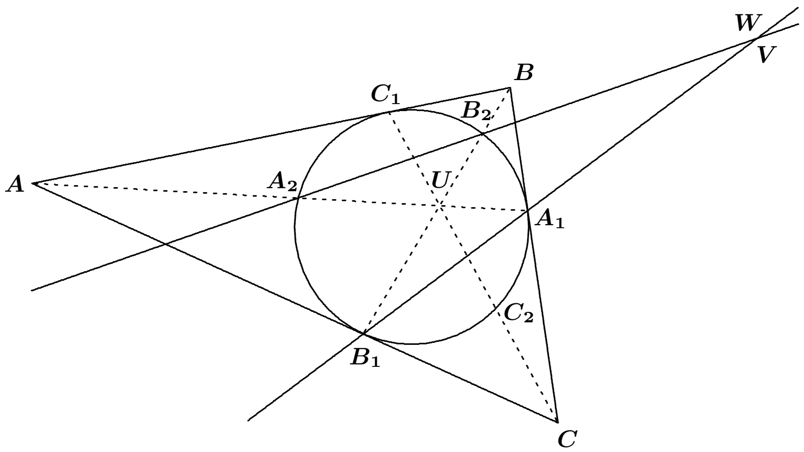
$$\begin{aligned} g(x+y) &= f(x+y) - (x+y)^2 \\ &= f(x) + f(y) + 2xy - (x+y)^2 \;=\; g(x) + g(y). \end{aligned}$$

This is the well-known Cauchy Equation, whose solutions are $g(x) = cx$, where c is a constant. Hence, $f(x) = x^2 + 2001x$ (since $f(1) = 2002$).

Readers' solutions to some of the problems from the 38th Mongolian Mathematical Olympiad, given in [2005 : 505], were presented in the March issue of the *Corner* [2007 : 86–88]. Next we look at solutions to two problems not discussed there.

3. The incircle of triangle ABC with $AB \neq BC$ touches sides BC and AC at points A_1 and B_1, respectively. The segments AA_1 and BB_1 meet the incircle at A_2 and B_2, respectively. Prove that the lines AB, A_1B_1, and A_2B_2 are concurrent.

Solution by Michel Bataille, Rouen, France.



Let the incircle Γ touch the side AB at C_1 and let $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$. Since

$$\frac{AB_1}{B_1C} \cdot \frac{CA_1}{A_1B} \cdot \frac{BC_1}{C_1A} \;=\; \frac{s-a}{s-c} \cdot \frac{s-c}{s-b} \cdot \frac{s-b}{s-a} \;=\; 1\,,$$

Ceva's Theorem shows that the lines AA_1, BB_1, and CC_1 are concurrent, say at U (U is the Gergonne Point of the triangle).

Let V be the pole of the line CC_1 with respect to the circle Γ. Since CA_1 and CB_1 are tangent to Γ at A_1 and B_1, respectively, the polar of C with respect to Γ is the line A_1B_1. By polar reciprocity, V is on A_1B_1. Similarly, the polar of C_1 is the line AB; hence, V is on AB. Now, let A_1B_1 and A_2B_2 meet at W. Since A_1A_2 and B_1B_2 meet at U, the polar of W with respect to Γ passes through U. But this polar also passes through C (since W is on A_1B_1). Thus, the polar of W is $CU = CC_1$ and $W = V$. Finally, V is on AB, A_1B_1, and A_2B_2 and the result follows.

6. Let A_1, B_1, and C_1 be the respective mid-points of the sides BC, AC, and AB of triangle ABC. Take a point K on the segment C_1A_1 and a point L on the segment A_1B_1 such that

$$\frac{C_1K}{KA_1} \;=\; \frac{BC+AC}{AC+AB} \qquad \text{and} \qquad \frac{A_1L}{LB_1} \;=\; \frac{AC+AB}{AB+BC}\,.$$

Let $S = BK \cap CL$. Show that $\angle C_1A_1S = \angle B_1A_1S$.

Solution by Michel Bataille, Rouen, France.

As usual, let $a = BC$, $b = CA$, and $c = AB$. Denote by $d(X, YZ)$ the distance from point X to the line YZ and by $[XYZ]$ the area of triangle XYZ. Since S is interior to $\triangle A_1B_1C_1$, the desired conclusion is successively equivalent to

$$\begin{aligned} S \text{ is on the internal bisector of } \angle B_1A_1C_1\,, & \\ d(S, A_1C_1) \;&=\; d(S, A_1B_1)\,, \\ A_1B_1 \cdot A_1C_1 \cdot d(S, A_1C_1) \;&=\; A_1C_1 \cdot A_1B_1 \cdot d(S, A_1B_1)\,, \\ c \cdot [SA_1C_1] \;&=\; b \cdot [SA_1B_1]\,. \end{aligned} \tag{1}$$

Denote by $\overrightarrow{X}$ the vector to X from a fixed origin. From the hypotheses, we have

$$\begin{aligned} (a+2b+c)\overrightarrow{K} \;&=\; (a+b)\overrightarrow{A}_1 + (b+c)\overrightarrow{C}_1 \\ (a+b+2c)\overrightarrow{L} \;&=\; (a+c)\overrightarrow{A_1} + (b+c)\overrightarrow{B}_1 \end{aligned}$$

$$\overrightarrow{B} \;=\; \overrightarrow{A}_1 - \overrightarrow{B}_1 + \overrightarrow{C}_1 \qquad \text{and} \qquad \overrightarrow{C} \;=\; \overrightarrow{A}_1 + \overrightarrow{B}_1 - \overrightarrow{C}_1\,.$$

Thus,

$$(a+2b+c)\overrightarrow{K} - b\overrightarrow{B} = (a+b+2c)\overrightarrow{L} - c\overrightarrow{C} = a\overrightarrow{A}_1 + b\overrightarrow{B}_1 + c\overrightarrow{C}_1\,;$$

whence (since $S = BK \cap CL$),

$$(a+b+c)\overrightarrow{S} = a\overrightarrow{A}_1 + b\overrightarrow{B}_1 + c\overrightarrow{C}_1\,.$$

As a result, $a:b:c = [SB_1C_1]:[SC_1A_1]:[SA_1B_1]$, and (1) follows.

Note: Since $B_1C_1 = \frac{1}{2}a$, $C_1A_1 = \frac{1}{2}b$, and $A_1B_1 = \frac{1}{2}c$, the result just obtained even shows that S is the incentre of $\triangle A_1B_1C_1$.

Also in the March *Corner* were some readers' solutions to problems of the 19th Balkan Mathematical Olympiad, given in [2005 : 506]. For these solutions, see [2007 : 88–90]. We now present another solution.

2. The sequence $a_1, a_2, \ldots, a_n, \ldots$ is defined by

$$a_1 = 20\,, \qquad a_2 = 30\,, \qquad a_{n+2} = 3a_{n+1} - a_n\,, \quad \text{for } n > 1.$$

Find all positive integers n for which $1 + 5a_na_{n+1}$ is a perfect square.

Solution by Michel Bataille, Rouen, France.

Since $a_3 = 70$ and $a_4 = 180$, we have $1 + 5a_3a_4 = 63001 = 251^2$. Thus, $n = 3$ is a solution. We show that there is no other solution.

Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, and let $\{F_n\}$ be the Fibonacci sequence, given by $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, $(n = 0, 1, 2, \ldots)$. Since the solutions to the equation $x^2 - 3x + 1 = 0$ are α^2 and β^2, the classical method easily leads to

$$a_n = 10\beta^2\alpha^{2n} + 10\alpha^2\beta^{2n} = 10(\alpha^{2n-2} + \beta^{2n-2})\,,$$

where the latter equality follows from $\alpha\beta = -1$. Using this, we see that

$$1 + 5a_na_{n+1} = 501 + (50F_{2n-1})^2\,.$$

Now, if this integer is a perfect square, say K^2, we have

$$501 = (K - 50F_{2n-1})(K + 50F_{2n-1})\,.$$

Thus, either $K - 50F_{2n-1} = 3$ and $K + 50F_{2n-1} = 167$ or $K - 50F_{2n-1} = 1$ and $K + 50F_{2n-1} = 501$. The first case yields $25F_{2n-1} = 41$, which is clearly impossible. The second case gives $F_{2n-1} = 5$, which implies that $n = 3$. This completes the proof.

An eagle-eyed reader has pointed out a slight oversight in the remark given with the solution to problem 3 of the Bulgarian Mathematical Olympiad [2005 : 506–507] discussed at [2007 : 89–90].

Comment by Daniel Tsai, student, Taipei American School, Taipei, Taiwan, modified by the editor.

In the solution to problem 3 of the Bulgarian Mathematical Olympiad, Final Round, 2003, given in the March 2007 issue of ***CRUX with MAYHEM***, is the remark that $x_n = f_{2n+1}$ for all $n \geq 1$. But for $n = 1$, we have $x_1 = 1 \neq 2 = f_3$. The remark should have stated that $x_n = f_{2n-3}$ for $n \geq 2$.

Now we turn to our file of solutions from our readers to problems given in the October 2006 issue of the *Corner*. We begin with solutions to problems of the First Round of the Iranian Mathematical Olympiad given at [2006 : 372].

1. Find all permutations $(a_1, \ldots, a_n)$ of $(1, \ldots, n)$ which have the property that $i + 1$ divides $2(a_1 + \cdots + a_i)$ for every i, $1 \leq i \leq n$.

Solution par Pierre Bornsztein, Maisons-Laffitte, France.

Pour $n = 1$, il n'existe évidemment qu'une seule permutation adéquate. Nous allons prouver que, pour tout $n \geq 2$, il existe exactement deux telles permutations, qui sont $(1, 2, 3, 4, \ldots, n)$ et $(2, 1, 3, 4, \ldots, n)$.

On vérifie directement que c'est le cas pour $n = 2$ et $n = 3$. De plus, il est facile de vérifier que ces deux permutations ont bien la propriété demandée pour tout $n \geq 2$.

Supposons que l'affirmation soit vraie pour $n - 1 \geq 2$. On considère une permutation $(a_1, a_2, \ldots, a_n)$ de $(1, 2, \ldots, n)$ ayant la propriété requise par l'énoncé. On va prouver que $a_n = n$. Alors l'hypothèse de récurrence assurera que $(a_1, a_2, \ldots, a_{n-1})$ est $(1, 2, 3, \ldots, n-1)$ ou $(2, 1, 3, \ldots, n-1)$.

On sait que $2(a_1 + a_2 + \cdots + a_n) = 2(1 + 2 + \cdots + n) = n(n+1)$. Aussi, d'après la propriété de l'énoncé, on sait que n divise $2(a_1 + a_2 + \cdots + a_{n-1})$. Par suite, n divise $2a_n$.

Cas 1. Si n est impair, il vient immédiatement que n divise a_n. Et, comme $a_n \in \{1, \ldots, n\}$, c'est donc que $a_n = n$.

Cas 2. Si $n = 2k$, on doit avoir a_n divisible par k. Si $a_n \neq n$, comme $a_n < 2k$, c'est donc que $a_n = k$. Mais $n - 1 = 2k - 1$ divise

$$\begin{aligned} 2(a_1 + a_2 + \cdots + a_{n-2}) &= n(n+1) - 2a_n - 2a_{n-1} \\ &= 2k(2k+1) - 2k - 2a_{n-1} \,. \end{aligned}$$

Comme $2k - 1$ est impair, on en déduit que $2k - 1$ divise

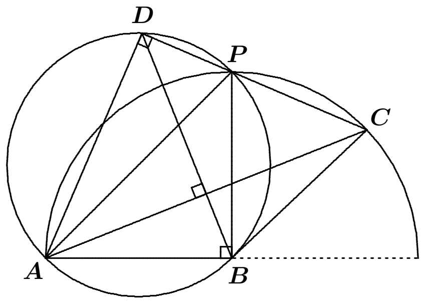
$$k(2k+1) - k - a_{n-1} \equiv k - a_{n-1} \pmod{2k-1} \,.$$

Puisque $a_{n-1} \in \{1, \ldots, 2k\}$, on a $-k \leq k - a_{n-1} \leq k - 1$. La seule possibilité d'avoir un multiple de $2k - 1$ est donc que $k - a_{n-1} = 0$, ou encore $a_{n-1} = k = a_n$, ce qui est impossible. Donc, on a bien $a_n = n$ et cela achève la démonstration.

4. Let A and B be two fixed points in the plane. Let $ABCD$ be a convex quadrilateral such that $AB = BC$, $AD = DC$, and $\angle ADC = 90^\circ$. Prove that there is a fixed point P such that, for every such quadrilateral $ABCD$ on the same side of the line AB, the line DC passes through P.

Solution by Michel Bataille, Rouen, France, modified by the editor.

On the same side of AB as the quadrilateral $ABCD$, draw the semi-circle (S) with centre B and radius BA, and the ray (R) originating at B and perpendicular to BA. We will show that all the lines CD pass through the point of intersection of (S) and (R).



Let $ABCD$ be an arbitrary quadrilateral satisfying the given conditions, and let P be the point of intersection of CD and (R).

We complete the proof by showing that this point P is on (S); that is, $BP = BC$. Let $\angle(\ell, \ell')$ denote the directed angle of the lines ℓ and ℓ'. Our goal will be reached if we prove the equality $\angle(PB, PC) = \angle(CP, CB)$.

We will use the fact that $\triangle ADC$ is right-angled and isosceles and that A, D, P, and B are concyclic (on the circle with diameter AP). Note also that BD is perpendicular to AC (since $BA = BC$ and $DA = DC$).

First, $\angle(PB, PC) = \angle(PB, PD) - \pi = \angle(AB, AD)$. Then,

$$\begin{aligned} \angle(CP, CB) &= \angle(CP, CA) + \angle(CA, CB) \\ &= \angle(AC, AD) + \angle(AB, AC) \\ &\qquad \text{(since } AD = DC \text{ and } AB = BC\text{)} \\ &= \angle(AB, AD)\,, \end{aligned}$$

and the result follows.

5. Let δ be a symbol such that $\delta \neq 0$ and $\delta^2 = 0$. Define

$$\begin{aligned} \mathbb{R}[\delta] &= \{a + b\delta \mid a\,,\, b \in \mathbb{R}\} \\ a + b\delta = c + d\delta &\iff a = c \text{ and } b = d\,, \\ (a + b\delta) + (c + d\delta) &= (a + c) + (b + d)\delta\,, \\ (a + b\delta) \cdot (c + d\delta) &= ac + (ad + bc)\delta\,. \end{aligned}$$

Let $P(x)$ be a polynomial with real coefficients. Show that $P(x)$ has a multiple root in $\mathbb{R}$ if and only if $P(x)$ has a non-real root in $\mathbb{R}[\delta]$.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's version.

Let a, $b \in \mathbb{R}$. An easy induction shows that for all $n \in \mathbb{N}$, we have $(a + b\delta)^n = a^n + na^{n-1}b\delta$. It follows that

$$P(x + y\delta) \;=\; P(x) + P'(x)y\delta \qquad (1)$$

for all real x and y.

If $P(x)$ has a multiple root x_0 in $\mathbb{R}$, then $P(x_0) = P'(x_0) = 0$ and, from (1), we have $P(x_0 + \delta) = 0$. Thus, $P(x)$ has a non-real root in $\mathbb{R}[\delta]$.

Conversely, if $P(a + b\delta) = 0$ for some real numbers a and b with $b \neq 0$, then, from (1) again, $P(a) + P'(a)b\delta = 0 = 0 + 0\delta$. Hence, $P(a) = P'(a) = 0$ and a is a multiple real root of $P(x)$.

6. Let G be a simple graph with 100 edges on 20 vertices. We can choose a pair of disjoint edges in 4050 ways. Prove that G is regular.

Solution par Pierre Bornsztein, Maisons-Laffitte, France.

Soient $V_1, \ldots, V_{20}$ les sommets de G, de degrés respectifs $d_1, \ldots, d_{20}$. Il s'agit de prouver que $d_1 = \cdots = d_{20}$.

Or, on sait que la somme des degrés est le double du nombre d'arêtes, donc

$$\sum_{i=1}^{20} d_i \;=\; 200\,. \qquad (1)$$

Soit (V_i, V_j) une arête. Il y a exactement $100-(d_i+d_j-1) = 101-(d_i+d_j)$ arêtes disjointes de (V_i, V_j). Et donc autant de paires d'arêtes disjointes dont une est (V_i, V_j). En sommant sur l'ensemble des arêtes, on obtient ainsi le double du nombre de paires d'arêtes disjointes (chacune est obtenue deux fois dans le raisonnement précédent).

Il vient donc $\sum\limits_{(V_i,V_j)} [101 - (d_i + d_j)] = 2 \times 4050$, ou encore

$$\sum_{(V_i,V_j)} (d_i + d_j) \;=\; 101 \times 100 - 2 \times 4050 \;=\; 2000\,.$$

Or, dans la somme ci-dessus, chaque d_i apparaît autant de fois qu'il existe d'arêtes dont un sommet est V_i, soit donc exactement d_i fois. Par conséquent, on a

$$2000 \;=\; \sum_{(V_i,V_j)} (d_i + d_j) \;=\; \sum_{i=1}^{20} d_i^2\,.$$

Mais, d'après (1) et l'inégalité entre les moyennes arithmétiques et quadratiques (AM/QM), on a alors

$$2000 \;=\; \sum_{i=1}^{20} d_i^2 \;\geq\; \frac{1}{20}\left(\sum_{i=1}^{20} d_i\right)^2 \;=\; 2000\,.$$

On est donc dans un cas d'égalité de AM/QM, ce qui signifie que tous les d_i sont égaux, comme désiré.

Next we look at solutions from our readers to problems of the Second Round of the Iranian Mathematical Olympiad 2002 given at [2006 : 373].

1. The sequence $\{a_n\}$ is defined by $a_0 = 2$, $a_1 = 1$, and $a_{n+1} = a_n + a_{n-1}$ for $n \geq 1$. Show that if p is a prime factor of $a_{2k} - 2$, then p is a factor of $a_{2k+1} - 1$.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We first give Bataille's exposition, followed by Bornsztein's.

The sequence $\{a_n\}$ is the Lucas sequence, often associated with the Fibonacci sequence $\{f_n\}$ defined by $f_0 = 0$, $f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$. As is well known, $a_n = \alpha^n + \beta^n$ and $f_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$, for $n = 0, 1, 2, \ldots$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Note that $\alpha + \beta = 1$ and $\alpha\beta = -1$. From these results, the following formulas are readily deduced:

$$a_{4n+2} - 2 = a_{2n+1}^2\,, \tag{1}$$
$$a_{4n+3} - 1 = a_{2n+2}a_{2n+1}\,, \tag{2}$$
$$a_{4n} - 2 = 5f_{2n}^2\,, \tag{3}$$
$$a_{4n+1} - 1 = 5f_{2n}f_{2n+1}\,. \tag{4}$$

Suppose first that k is odd, say $k = 2n + 1$ for some $n \geq 0$. If p is a prime factor of $a_{2k} - 2 = a_{4n+2} - 2$, then, from (1), p is a prime factor of a_{2n+1}^2 and hence of a_{2n+1}. Therefore, from (2), p is a prime factor of $a_{2k+1} - 1 = a_{4n+3} - 1$.

Similarly, if k is even, we deduce from (3) that a prime factor of $a_{2k} - 2$ is 5 or a prime factor of f_{2n}. In any case, as (4) shows, this prime factor divides $a_{2k+1} - 1$.

Nous donnons aussi l'approche de Bornsztein.

On pose $U_n = a_{n-1}a_{n+1} - a_n^2$. On a directement $U_1 = 5$. Et, pour tout $n \geq 1$, il vient :

$$\begin{aligned} U_{n+1} &= a_n a_{n+2} - a_{n+1}^2 = a_n a_{n+1} + a_n^2 - a_{n+1}^2 \\ &= a_{n+1}(a_n - a_{n+1}) + a_n^2 = a_{n+1}(-a_{n-1}) + a_n^2 = -U_n\,. \end{aligned}$$

Par conséquent, pour tout $n \geq 1$, on a $U_n = 5(-1)^{n-1}$, ou encore

$$a_{n-1}a_{n+1} - a_n^2 = 5(-1)^{n-1}\,. \tag{1}$$

Soit p un nombre premier qui divise $a_{2k} - 2$; donc $a_{2k} \equiv 2 \pmod{p}$. D'après (1), on a $a_{2k}^2 - 5 = a_{2k-1}a_{2k+1} = (a_{2k+1} - a_{2k})a_{2k+1}$, puis $-1 \equiv a_{2k+1}^2 - 2a_{2k+1} \pmod{p}$, ou encore $(a_{2k+1} - 1)^2 \equiv 0 \pmod{p}$. Ainsi, p divise $a_{2k+1} - 1$, comme désiré.

2. Let A be a point outside the circle Ω. The tangents from A to Ω touch Ω at B and C. A tangent L to Ω intersects AB and AC at P and Q, respectively. The line parallel to AC passing through P meets BC at R. Prove that as L varies, QR passes through a fixed point.

Comment by Michel Bataille, Rouen, France.

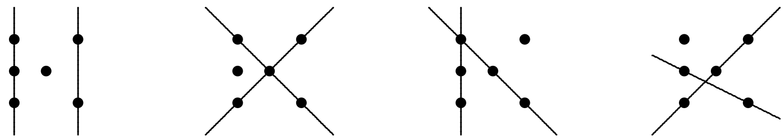
This problem is not new. It is problem 2639 ([2001 : 268; 2002 : 272]). Two different solutions were given in the May 2002 issue.

4. Find the smallest positive integer n for which the following condition holds: For every finite set of points in the plane, if, for every n points in this set, there exist two lines covering all n points, then there exist two lines covering all points in the set.

Solution par Pierre Bornsztein, Maisons-Laffitte, France.

Le plus petit entier n cherché est $n = 6$.

Soit $\mathcal{E}_5$ l'ensemble formé des points $A(0,0)$, $B(2,0)$, $C(2,2)$, $D(0,2)$, $M(0,1)$ et $\Omega(1,1)$. Les schémas suivant montrent que toute partie à 5 éléments de $\mathcal{E}_5$ peut être recouverte par deux droites.



Par contre, $\mathcal{E}_5$ ne peut être recouvert par deux droites : en effet, comme $\mathcal{E}_5$ ne contient pas quatre points alignés, tout recouvrement éventuel de $\mathcal{E}_5$ par deux droites se ferait par deux droites contenant chacune trois points. Le seul groupe de trois points alignés contenant M est A, M, D. Ainsi, c'est nécessairement l'un des deux groupes qui définit une des deux droites. L'autre doit alors passer par les trois points restant, mais ceux-ci ne sont pas alignés. Cela prouve que $n = 5$ n'a pas la propriété désirée.

On prouve maintenant $n = 6$ possède la propriété de l'énoncé. Cela étant, il convient d'intérpréter cette condition en :

> For every finite set of at least n points in the plane, if, for every n points in this set, there exist two lines covering all n points, then there exist two lines covering all points in the set.

Sinon, il n'existe aucun entier vérifiant la condition demandée (prendre l'ensemble des sommets d'un pentagone régulier. Pour $n \geq 6$, il n'existe aucun ensemble de n points dans cet ensemble donc les prémisses sont trivialement vérifiées, mais l'ensemble ne peut être recouvert par deux droites).

Soit $\mathcal{E}$ un ensemble d'au moins 6 points tel que tout sous-ensemble de 6 points de $\mathcal{E}$ puisse être recouvert par deux droites.

Soit $\mathcal{F} \subset \mathcal{E}$, avec $|\mathcal{F}| = 6$. Comme $\mathcal{F}$ peut être recouvert par deux droites, le principe des tiroirs assure qu'au moins trois des points de $\mathcal{F}$, et donc de $\mathcal{E}$, sont alignés. Disons que A, B, C sont trois points, deux à deux distincts, alignés et appartenant à $\mathcal{E}$. On note Δ la droite (AB).

Si $\mathcal{E} - \Delta$ ne contient pas plus de deux points, alors $\mathcal{E}$ peut clairement être recouvert par deux droites. Si $\mathcal{E} - \Delta$ contient au moins trois éléments : Soient X et Y deux points distincts dans $\mathcal{E} - \Delta$. On note Δ' le droite (XY).

Pour tout point $M \in \mathcal{E} - \Delta$, autre que X et Y, l'ensemble formé des points A, B, C, X, Y, M peut être recouvert par deux droites. Si aucune de ces deux droites n'est Δ, celle qui recouvre A ne recouvre ni B ni C. Donc l'autre doit recouvrir à la fois B et C, mais alors c'est Δ, une contradiction.

Ainsi, l'une des deux droites est Δ. l'autre doit nécessairement recouvrir les points X, Y et M, ce qui prouve que $M \in \Delta'$.

Par conséquent, tout point de $\mathcal{E}$ appartient à Δ ou à Δ', et donc $\mathcal{E}$ peut être recouvert par deux droites.

6. Let a, b, and c be positive real numbers such that $a^2+b^2+c^2+abc=4$. Prove that $a+b+c \leq 3$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Vedula N. Murty, Dover, PA, USA. We first give the solution of Díaz-Barrero, using a classic change of variables and geometry.

Setting $a = 2\cos A$, $b = 2\cos B$, and $c = 2\cos C$, with $A+B+C = \pi$, we have

$$\begin{aligned} & a^2+b^2+c^2+abc \\ &= 4\cos^2 A + 4\cos^2 B + 4\cos^2(A+B) - 8\cos A\cos B\cos(A+B) \\ &= 4\cos^2 A + 4\cos^2 B - 4\cos^2 A\cos^2 B + 4\sin^2 A\sin^2 B \\ &= 4\sin^2 B(\cos^2 A + \sin^2 A) + 4\cos^2 B \;=\; 4\,. \end{aligned}$$

Taking into account Euler's Inequality, $R \geq 2r$, and the well-known identity $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$, we get

$$a+b+c \;=\; 2(\cos A + \cos B + \cos C) \;=\; 2\left(1+\frac{r}{R}\right) \;\leq\; 3\,.$$

Note that equality holds when $a = b = c = 1$.

Next we give the solution of Murty.

Without loss of generality, we assume that $0 < a \leq b \leq c$. From $a^2+b^2+c^2+abc = 4$ we deduce that $0 < a \leq 1$, $0 < b < 2$, and $1 \leq c < 2$. Now $c^2 + c(ab) + a^2 + b^2 - 4 = 0$ is a quadratic equation in c and the positive root is

$$c \;=\; \tfrac{1}{2}\left(-ab + \sqrt{(4-a^2)(4-b^2)}\right).$$

Hence, $a+b+c \leq 3$ if and only if

$$\sqrt{(4-a^2)(4-b^2)} \;\leq\; 6 - 2a - 2b + ab\,. \qquad (1)$$

From the AM–GM Inequality, we have

$$\sqrt{(4-a^2)(4-b^2)} \;\leq\; \tfrac{1}{2}\left(8 - (a^2+b^2)\right).$$

We now prove that

$$\tfrac{1}{2}\big(8-(a^2+b^2)\big) \;\le\; 6-2a-2b+ab\,. \tag{2}$$

This inequality is equivalent to $(a+b)^2-4(a+b)+4\ge 0$, which factors as $(a+b-2)^2\ge 0$. Thus (2) is true. Then (1) is true. Equality is attained when $a=b=c=1$.

Finally, we look at solutions from our readers to problems of the Third Round of the Iranian Mathematical Olympiad 2002 given in [2006 : 373–374].

1. Find all real polynomials $P(x)$ such that $P(a)\in\mathbb{Z}$ implies that $a\in\mathbb{Z}$.

Solution par Pierre Bornsztein, Maisons-Laffitte, France.

Les solutions sont les polynômes constants et ceux de la forme $P(x)=(x+b)/c$, où b et c sont deux entiers avec $c\ne 0$.

Il est facile de vérifier que les polynômes ci-dessus conviennent effectivement.

Soit P un polynôme non constant ayant la propriété de l'énoncé. Quitte à changer P en $-P$, on peut supposer que

$$\lim_{x\to+\infty} P(x) \;=\; +\infty\,. \tag{1}$$

Cas 1. On suppose que P est de degré au moins égal à 2.

D'après (1), on a alors

$$\lim_{x\to+\infty} P'(x) \;=\; +\infty\,. \tag{2}$$

D'après (1) et (2), et puisqu'il s'agit de polynômes, il existe donc $A>0$ tel que P et P' soient strictement croissants sur $[A,+\infty)$. Toujours d'après (2), on peut également supposer que

$$P'(x)>2 \text{ pour } x\ge A. \tag{3}$$

Soit q un entier tel que $q\ge P(A)$. D'après la propriété de l'énoncé et le théorème des valeurs intermédiaries, pour tout entier $n\ge 0$, il existe un entier $x_n\ge A$ tel que

$$P(x_n) \;=\; q+n\,. \tag{4}$$

Puisque P est strictement croissant sur $[A,+\infty)$, la suite $\{x_n\}$ est donc strictement croissante. En particulier, s'agissant d'entiers, on a

$$x_{n+1}-x_n \;\ge\; 1 \quad \text{pour tout } n\ge 0. \tag{5}$$

D'après le théorème des accroissements finis, pour tout entier $n\ge 0$, il existe $y_n\in[x_n,x_{n+1}]$ tel que $P(x_{n+1})-P(x_n)=(x_{n+1}-x_n)P'(y_n)$.

D'après (3) et (5), on a donc $P(x_{n+1}) - P(x_n) > 2$. Mais, d'après (4), on a $P(x_{n+1}) - P(x_n) = 1$, une contradiction. Il n'existe donc aucun polynôme de degré supérieur ou égal à 2 possédant la propriété désirée.

Cas 2. On suppose que $P(x) = \alpha x + \beta$, où α et β sont des réels et $\alpha > 0$. Alors, d'après la propriété de l'énoncé, ils existent deux entiers x_0 et x_1 tels que $P(x_0) = 0$ et $P(x_1) = 1$. Donc $\alpha x_0 + \beta = 0$ et $\alpha x_1 + \beta = 1$, d'où $\alpha(x_1 - x_0) = 1$. On a $x_1 - x_0 > 0$, puisque $\alpha > 0$. Soit $c = x_1 - x_0$. Alors $\alpha = 1/c$ et $\beta = -x_0/c$, ce qui prouve que P est bien de la forme annoncé et achève la démonstration.

3. In a triangle ABC, define C_a to be the circle tangent to AB, to AC, and to the incircle of the triangle ABC, and let r_a be the radius of C_a. Define r_b and r_c in the same way. Prove that $r_a + r_b + r_c \geq 4r$, where r is the inradius of the triangle ABC.

Solution by Michel Bataille, Rouen, France.

The given inequality is false (for example $r_a = r_b = r_c = \frac{r}{3}$ in an equilateral triangle). We will prove instead that

$$r_a + r_b + r_c \geq r . \tag{1}$$

The circle C_a is the image of the incircle in the homothety with centre A and scale factor r_a/r. Hence, if I is the incentre and I_a is the centre of C_a, we have $\overrightarrow{AI_a} = \frac{r_a}{r}\overrightarrow{AI}$, which can be rewritten as

$$\overrightarrow{II_a} = \left(\frac{r_a}{r} - 1\right)\overrightarrow{AI} .$$

Since $r_a < r$ and $II_a = r + r_a$, it follows that

$$\frac{r_a}{r} = \frac{AI - r}{AI + r} = \frac{2AI}{AI + r} - 1 = \frac{2}{1 + (r/AI)} - 1 = \frac{2}{1 + \sin(A/2)} - 1 .$$

Similar results hold for r_b and r_c. Thus, we see that (1) is equivalent to

$$\frac{1}{1+\sin(A/2)} + \frac{1}{1+\sin(B/2)} + \frac{1}{1+\sin(C/2)} \geq 2 . \tag{2}$$

The function $f(x) = 1/(1+\sin x)$ is strictly convex on $\left(0, \frac{\pi}{2}\right)$ (its second derivative is $f''(x) = (1+\sin x)^{-3}(1 + \sin x + \cos^2 x)$, which is positive); hence,

$$f\left(\frac{A}{2}\right) + f\left(\frac{B}{2}\right) + f\left(\frac{C}{2}\right) \geq 3f\left(\frac{A+B+C}{3}\right) = 3f\left(\frac{\pi}{6}\right) = 2 .$$

Therefore, (2) holds. Equality is attained in (2) if and only if $A = B = C$; that is, if and only if $\triangle ABC$ is equilateral.

That completes the material for this number of the *Corner*. Please send your nice generalizations and solutions as well as Olympiad Contests.

BOOK REVIEWS

John Grant McLoughlin

First Steps for Math Olympians: Using the American Mathematics Competitions
By J. Douglas Faires, Mathematical Association of America, 2006
ISBN 0-88385-824-X, hardcover, 307+xxii pages, US$46.50
Reviewed by ***Robert D. Poodiack****, Norwich University, Northfield, Vermont, USA*

This excellent book adds to the problem-solving literature not only as a source of problems and solutions, but also as a primer of concepts and techniques for problem-solving. The book is limited, however, to pre-calculus mathematics.

In the introduction, Professor Faires writes about the history of, and his own involvement with, the American High School Mathematics Examination. This examination, now split into grade levels as part of the American Mathematics Competition (AMC), has been widely used as a springboard for students toward taking the American Invitational Mathematics Examination (AIME). Students who do well enough on the AIME may be asked to participate in the United States of America Mathematical Olympiad and the International Mathematical Olympiad.

The book is split into 18 chapters, each dealing with a single topic. The topics range through arithmetic and algebra, functions and geometry, sequences and series, probability and statistics, and trigonometry and basic number theory. Most chapters begin with an enumeration of basic definitions and results. Few formal proofs are included, most likely for brevity. For many results, though, an idea of why the result is true is included. As the target audience is high school students, this is not a large problem.

In each chapter, major theorems are presented for drill purposes. By organizing the book according to topics rather than presenting old AMC exams in chronological order, the author has made this book easier to use than the various MAA olympiad question books (where important results are listed in alphabetical order in an appendix). The sheer number of results can be mind-boggling, especially in the geometry sections, but they will all be recognized as useful by anyone who has studied mathematics competition problem collections. (I had only a vague recollection of the Tangent Chord Theorem or the External Secant Theorem myself!)

The meat of the book is, of course, in the problems. All the problems in the book are taken from old AMC exams. Each chapter ends with three completely solved "examples," in increasing order of difficulty, followed by a set of 10 related exercises, whose answers are in the back of the book. As in the actual AMC exams, all of the examples and exercises are multiple-choice. The problems and solutions presented are quite stimulating and ingenious.

Topics worthy of student research are sometimes introduced in a disguised manner.

For instance, Exercise 8 in Chapter 12, on sequences and series, reads:

> Alice, Bob, and Carol repeatedly take turns tossing a fair regular six-sided die. Alice begins; Bob always follows Alice; Carol always follows Bob; and Alice always follows Carol. Find the probability that Carol will be the first to toss a six.

In addition to the solution involving an infinite geometric series, this example can lead to a discussion of probability "trees" that contain "wreaths" in their structure. If we change the probabilities of success for each person, we enter the theory of three-way duels!

In another example, in the chapter on functions, students are asked to find the maximum value of

$$f(x) \;=\; \sqrt{8x - x^2} - \sqrt{14x - x^2 - 48}\,.$$

The author presents a canny solution involving the geometry of semi-circles and, most notably, *no calculus*.

In his solution to the previous problem, Professor Faires makes a point of noting that "no problem on the AMC has a calculus solution that is easier than some non-calculus solution". Given the intended audience of high-school and middle-school students, the AMC exams and this book leave out calculus altogether, as well as other advanced topics. University competitors will want to augment this book with another more advanced competition book.

My only complaints about the book are organizational. Since the full solutions to all of the exercises are given in the back of the book, it would have been nice for the questions and answer choices to be reprinted with them. For old forgetful folks like me, some cross-referencing of page numbers between questions and answers would have been welcome.

These are mere quibbles, though. *First Steps for Math Olympians* is an excellent introductory primer for any precocious student gearing up for the AMC exam. Professor Faires has provided a well-organized, easily digested study guide appropriate for high-school and middle-school students as well as any college student just starting out in competitions. The volume and quality of problems make the book worthwhile for students and coaches alike. I plan on paying the ultimate compliment of filching some of these problems for a competition I organize this year!

The Edge of the Mathematical Universe:
Celebrating 10 Years of Math Horizons
Edited by Deanna Haunsperger and Stephen Kennedy, Mathematical Association of America, 2006
ISBN 0-88385-550-0, hardcover, xii+303 pages, US$57.50.
Reviewed by **John Grant McLoughlin**, *University of New Brunswick, Fredericton, NB.*

Wow! Some books arrive that have me saying, "I'll review that one!" This is one of those books. The collection of articles and the overall presentation of *The Edge of the Mathematical Universe* invite distraction from other activities as one engages with the mathematical playfulness evident on its pages. The book is a tribute to the MAA publication, *Math Horizons*. The editors, Deanna Haunsperger and Stephen Kennedy, guided *Math Horizons* through the second half of its first decade when they took over from founding editor Donald Albers. Here they organize a selection of approximately 75 articles chronologically ordered into a resource that belongs in the lounges and reading rooms of mathematics departments and mathematicians alike. The chronological ordering of articles may be displeasing to those who find the seemingly jumpy nature of the topics to be less than ideal. Personally, I found the presentation more like that of a Martin Gardner book which likewise typically consists of segments arising out of columns that may be seemingly unrelated to their immediate neighbours.

The authors (Guy, Gardner, Wagon, Dudley, Dunham, ...) will be familiar to any reader of recreational mathematics or other material geared to undergraduate or senior secondary mathematical audiences. Anyone reading this review is bound to enjoy the offerings of this book. There really is no simple way to capture its scope and breadth. Readers familiar with *Math Horizons* will have a sense of what to expect; the rest of you may wish to avail of a genuine mathematical smorgasbord of ideas. The table of contents spans four pages. The opening article, entitled "John Horton Conway—Talking a Good Game" (reprinted from Spring 1994), and the closing article, "Knots to You" (reprinted from November 2003), surround an assortment of others, including: "Weird Dice", "The Instability of Democratic Decisions", "Was Gauss Smart?", "Egyptian Rope, Japanese Paper and High School Math", and "The World's First Mathematics Textbook".

Rarely have I seen such a wonderful assortment of mathematics displayed in such an accessible manner. This book would be a rich resource for mathematics clubs, budding secondary school mathematicians, or anyone further along in their mathematical journey. This may become a core reference in one of my future courses with prospective or practicing mathematics teachers, as it offers plenty of content along with insight into what doing mathematics is really about.

The Converse of Schiffler's Theorem

Joe Goggins

The Schiffler point of a triangle is named after the proposer of problem 1018 [1985 : 51; 1986 : 150–152]. We have recently observed the twentieth anniversary of this notable discovery, and while the explosion of interest in the topic continues to amaze us, at least one simple aspect seems to have been overlooked. The situation is this: If P is a point in the plane of triangle ABC, but not on any of its side lines, then the Euler lines of the four triangles ABC, PBC, APC, and ABP may or may not concur. Kurt Schiffler discovered that *when P is located at the incentre of $\triangle ABC$, then the Euler lines concur at the point now bearing his name*. Let us use the notation found in Clark Kimberling's *Encyclopedia of Triangle Centers* (ETC) [3], where the incentre is denoted by X_1 and the Schiffler point by X_{21}. Schiffler's Theorem is then

$P = X_1$ implies the four Euler lines concur at X_{21}.

The converse, however, is false since there is a second valid solution that appears in ETC as X_{3065}:

The concurrence of the Euler lines at X_{21} implies that $P = X_1$ or $P = X_{3065}$.

Confirming that X_{3065} is defined by trilinear coordinates $x : y : z$, where

$$\begin{aligned} x &= \frac{1}{1+2(\cos A - \cos B - \cos C)}, \\ y &= \frac{1}{1+2(\cos B - \cos C - \cos A)}, \\ \text{and} \quad z &= \frac{1}{1+2(\cos C - \cos A - \cos B)}. \end{aligned}$$

gives impetus to further investigations. The point X_{3065} turns out to be the isogonal conjugate of X_{484} (the first Evans perspector), which we write as $X_{3065} = X_{484}^{-1}$. Bernard Gibert mentions the point X_{484}^{-1} on his website [2] as $E546$, but not in connection with X_{21} and the concurrence properties that are of interest here.

Although there is an elementary geometric proof of Schiffler's theorem in ***Crux Mathematicorum***, I have yet to find a comparable argument for the converse. However, since we know the trilinear coordinates of X_{3065} and X_{21}, we can find the equations of the Euler lines of ABX_{3065}, etc., and confirm algebraically that these Euler lines do indeed pass through X_{21}.

The point $X_{3065} = X_{484}^{-1}$ has a relatively simple construction, as indicated in Figure 1 (based on the description of X_{484} in ETC):

1. Locate the incentre I and the excentres I_A, I_B, and I_C of $\triangle ABC$.
2. Define A', B', and C' to be the reflected images of the vertices A, B, and C in the opposite sides BC, CA, and AB, respectively.
3. Define D to be the point of concurrence of lines $A'I_A$, $B'I_B$, and $C'I_C$ (concurrent specifically at point X_{484}, the first Evans perspector).
4. Define D_A, D_B, and D_C to be the reflections of D in AI, BI, and CI, respectively.
5. The required point X_{484}^{-1} is the point of concurrence of lines AD_A, BD_B, and CD_C (concurrent specifically at the isogonal conjugate of point X_{484}).

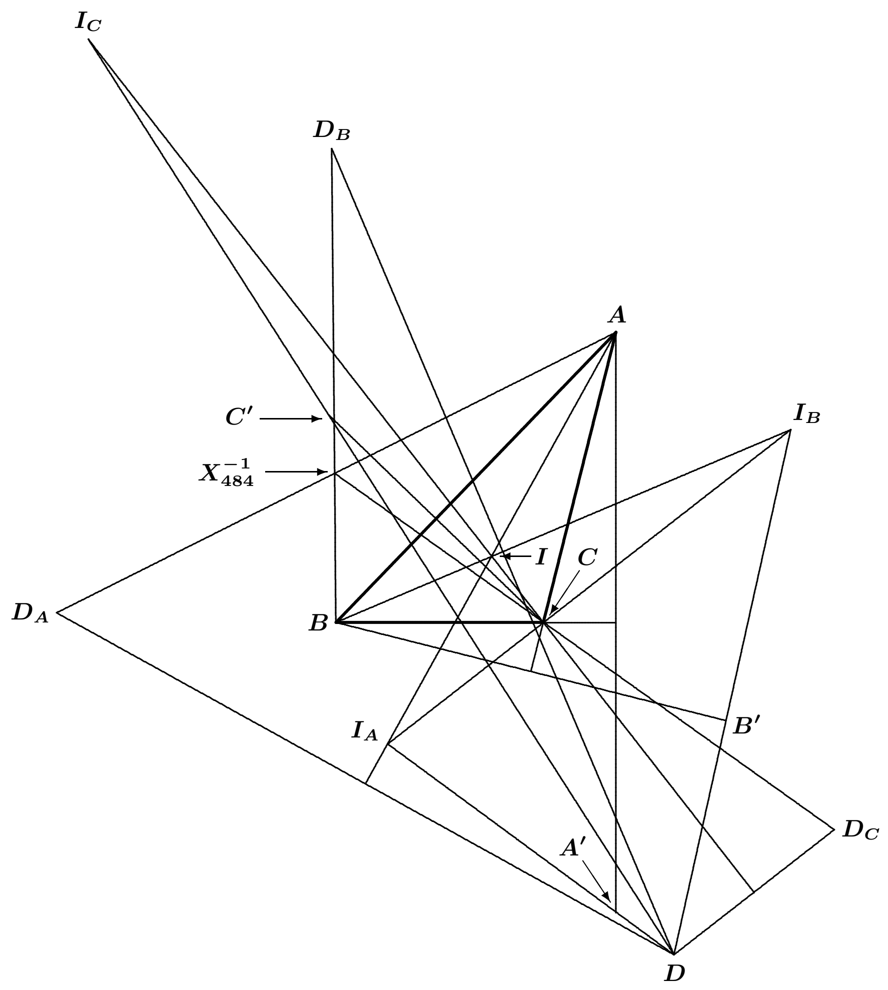


Figure 1: Construction of $X_{3065} = X_{484}^{-1}$

Not unexpectedly, our converse has a more general aspect. The investigation is guided by a theorem that appears as an exercise in [4], page 200; a proof can be found in [1], Theorem 6.1.

The locus of a point P in the plane of triangle ABC (with side lines omitted) such that the Euler lines of the four triangles ABC, PBC, APC, and ABP concur is the union of the Neuberg cubic of $\triangle ABC$ and its circumcircle.

The theorem does not, of course, give any guidance as to the location of the point of concurrence with respect to the given point P, but it does provide a good starting point for a computer investigation, assisted by the established data in ETC. Some numbered assertions follow. There is good computer evidence for them all, but I have verified only those found in the table in (3) below.

Notation. The Parry reflection point, X_{399}, lies on the Neuberg cubic (an established fact). A secant through this point, but not tangent to the cubic, will cut the cubic at two further points Z and Z', and the Euler line of $\triangle ABC$ at T, say. Let O be the circumcentre of $\triangle ABC$ and S the mid-point of OT. Then we have

(1) Conjecture: The Euler lines of the triangles ABC, ZBC, AZC, ABZ, $Z'BC$, $AZ'C$, and ABZ' concur at S.

We refer to Z and Z' as Schiffler conjugates; each position $P = Z$ or $P = Z'$ determines Euler lines that concur at the same point S.

(2) Conjecture: Every pair of Schiffler conjugates lies on a line through X_{399}.

Note that conjectures (1) and (2) combine to assert that, in general, each concurrence point S comes from precisely two positions of P on the Neuberg cubic. (Of course, if P is any point different from a vertex on the circumcircle of $\triangle ABC$ then the four resulting Euler lines all pass through the circumcentre.)

We now list some examples where Schiffler conjugates are found among centres that have been indexed in ETC (and therefore have a ready basis of attestation). Each entry amounts to an individual assertion. See Figure 2. Again we denote the isogonal conjugate of X_N by X_N^{-1}.

(3) Examples of Schiffler conjugates:

We refer to a point X_R in the table below. Its trilinear coordinates are $\alpha : \beta : \gamma$ where

$$\alpha = \frac{2U_2}{\cos B + \cos C} - U_1 \cos A\,, \qquad \beta = \frac{2U_2}{\cos C + \cos A} - U_1 \cos B\,,$$

$$\text{and} \qquad \gamma = \frac{2U_2}{\cos A + \cos B} - U_1 \cos C\,,$$

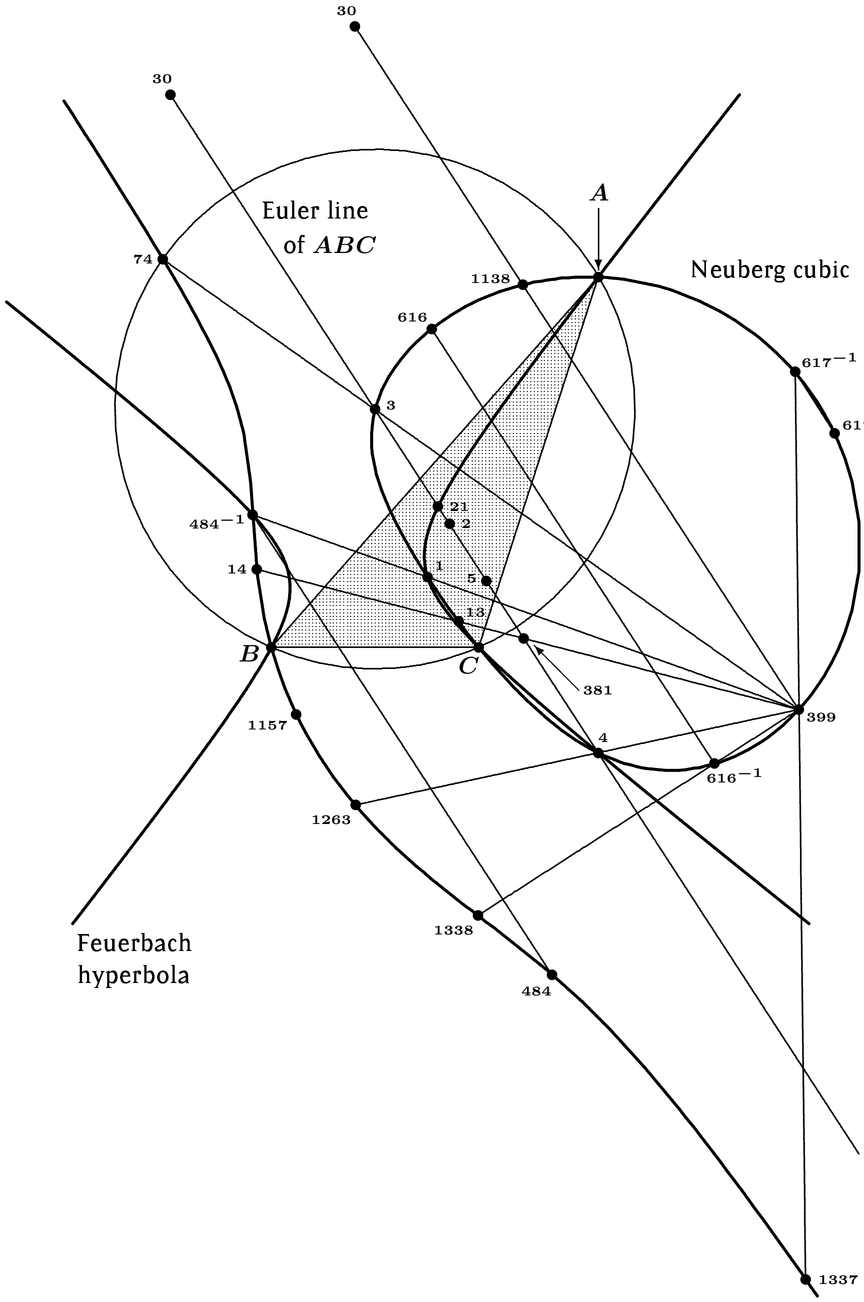


Figure 2: Neuberg cubic, Feuerbach hyperbola, and examples of Schiffler conjugates

with

$$\begin{aligned} U_1 &= \frac{a}{\cos B + \cos C} + \frac{b}{\cos C + \cos A} + \frac{c}{\cos A + \cos B} \\ \text{and} \quad U_2 &= a\cos A + b\cos B + c\cos C\,. \end{aligned}$$

Z	Z'	S = Concurrence point on Euler line of $\triangle ABC$	T = intersection of ZZ' with Euler line of $\triangle ABC$
X_1 (incentre)	$X_{3065} = X_{484}^{-1}$	X_{21} (Schiffler pt)	X_R
X_{13} (1st Fermat pt)	X_{14} (2nd Fermat pt)	X_2 (centroid)	X_{381} (mid-point of X_2 and X_4)
X_4 (orthocentre)	X_{1263}	X_5 (9-pt centre)	X_4 (orthocentre)
X_3 (circumcentre)	$X_{74} = X_{30}^{-1}$	X_3 (circumcentre)	X_3 (circumcentre)
$X_{1138} = X_{399}^{-1}$	X_{30} (Euler infinity pt)	X_{30} (Euler infinity pt)	X_{30} (Euler infinity pt)
X_{1337} (1st Wernau pt)	X_{617}^{-1}	S_1 (Not indexed in ETC)	X_3 reflected in S_1
X_{1338} (2nd Wernau pt)	X_{616}^{-1}	S_2 (Not indexed in ETC)	X_3 reflected in S_2

Let H be the orthocentre of $\triangle ABC$. Then, with reference to claim (2) above, we now have seven notable points (A, B, C, H, Z, Z', and S). Taking three points at a time (as vertices) we can, in general, form 35 distinct triangles. Each of these has a nine-point circle, four of which will be identical (for ABC, HBC, AHC, and ABH). This leaves up to 32 distinct circles in the plane of ABC, which I will refer to as a *Schiffler Set*.

(4) Conjecture: All circles of a Schiffler Set concur at a fixed point.

(5) Some examples (again, individual assertions):

1. For the Schiffler Set $(A, B, C, H, X_1, X_{3065}, X_{21})$, the point of concurrence is X_{11}.

2. For the Schiffler Set $(A, B, C, H, X_{13}, X_{14}, X_2)$, the point of concurrence is X_{115}.

3. For the Schiffler Set $(A, B, C, H, X_{1263}, X_4, X_5)$, the point of concurrence is X_{137}. (Note that $X_4 = H$.)

4. For the Schiffler Set $(A, B, C, H, X_3, X_{74}, X_3)$, the point of concurrence is X_{125}.

We have observed that in each case all seven points lie on a conic, the centre of which, call it X_C, happens to be the point of concurrence of the Schiffler Set that they define. For example, the conic through A, B, C, H, X_1, X_{3065}, and X_{21} is the Feuerbach hyperbola with centre $X_C = X_{11}$. This suggests a more comprehensive theorem:

(6) Conjecture: If point Z lies on the Neuberg cubic of ABC, then the hyperbola through A, B, C, H, and Z also passes through Z', on the Neuberg cubic, and S, the point of concurrence of the Euler lines of triangles ABC, ZBC, AZC, ABZ, $Z'BC$, $AZ'C$, and ABZ'. The centre of the hyperbola, X_C, is the point of concurrence of the Schiffler set.

Remark. Bernard Gibert, in his study of the Neuberg cubic [2] mentions the isogonal conjugates $X_{616}^{-1} = E558$ and $X_{617}^{-1} = E559$, but not in the above context.

Finally, there are elementary proofs for some of the entries in the table in (3). Here is a proof that $P = X_{13}$ and $P = X_{14}$ both determine Euler lines that intersect in the centroid.

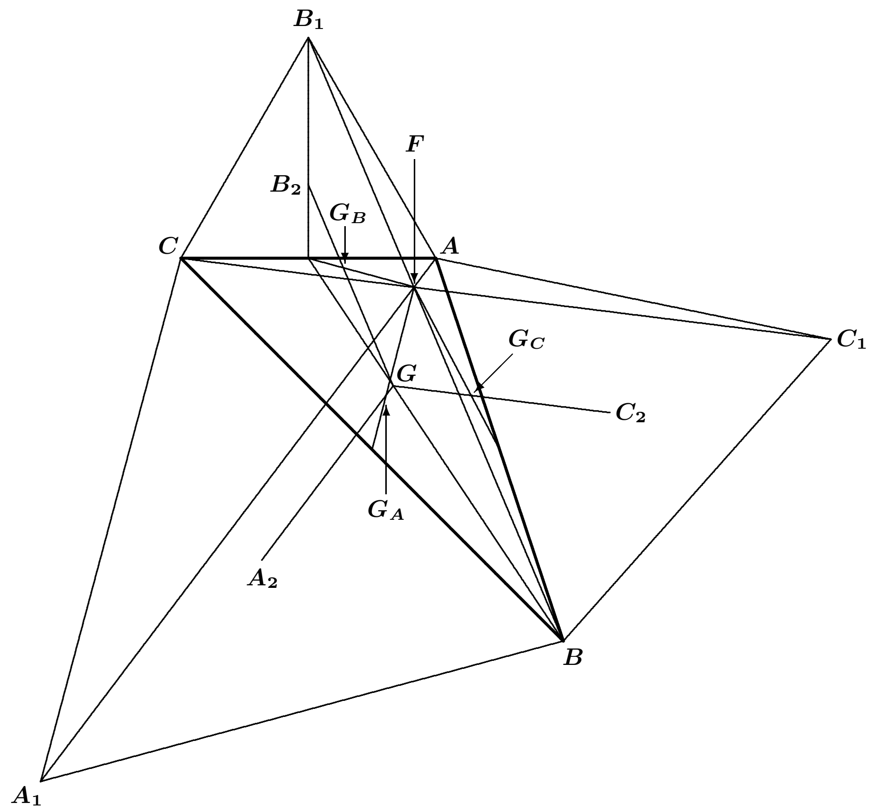


Figure 3: $F = X_{13}$ determines the Euler lines that meet at the centroid G

The first Fermat point $F = X_{13}$ is the intersection of the lines joining the vertices of $\triangle ABC$ to the remote vertices A_1, B_1, and C_1 of equilateral triangles erected externally on the opposite sides (see Figure 3). The centroids B_2 of $\triangle AB_1C$, G_B of $\triangle FAC$, and G of $\triangle BAC$ are each one third of the way from the mid-point of AC to the opposite vertices, which implies that B_2G_B and B_2G are parallel to B_1B (which contains F by definition). That is, G_B lies on B_2G. But B_2 is the circumcentre of $\triangle AFC$; thus, B_2G is the Euler line of $\triangle AFC$. Similarly, C_2G and A_2G are the Euler lines of triangles ABF and BCF. Therefore, point F effects the concurrence of the Euler lines at the centroid of $\triangle ABC$.

The same argument shows that the second Fermat point X_{14}, constructed using equilateral triangles erected internally on the sides of $\triangle ABC$, likewise determines Euler lines that meet at G.

The proof that $P = X_3$ and $P = X_{74}$ both determine Euler lines that intersect in the circumcentre X_3 is left to the reader. It is less complicated than the above proof.

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On the Pell Equation $x^2 - (k^2 - 2)y^2 = 2^t$

Ahmet Tekcan

1 Introduction.

Let $d \neq 1$ be a positive non-square integer and N be any fixed positive integer. Then the equation

$$x^2 - dy^2 = \pm N \tag{1.1}$$

is known as "Pell's equation" after John Pell (1611-1685), who searched for integer solutions to equations of this type. Ironically, Pell was not the first to work on this problem, nor did he contribute to our knowledge for solving it. Euler (1707-1783), who brought us the ϕ-function, named the equation after Pell, and the name stuck.

For $N = 1$, the Pell equation

$$x^2 - dy^2 = \pm 1 \tag{1.2}$$

is known as the *classical* Pell equation. Its complete theory was worked out by Lagrange (1736-1813), not Pell. It is often said that Euler mistakenly attributed Brouncker's (1620-1684) work on this equation to Pell. However, the equation appears in a book by Rahn (1622-1676), which was certainly written with Pell's help. Perhaps Euler knew what he was doing in naming the equation. Further details can be found in [2], [6], and [7].

In this article, we will define by recurrence an infinite sequence of positive solutions of the Pell equation $x^2 - dy^2 = 2^t$, where $d = k^2 - 2$ with $k > 2$ an integer and $t \geq 0$ is also an integer. We will also express these solutions using matrices that depend only on k and t.

2 Preliminary facts.

The Pell equation in (1.2) has infinitely many integer solutions. The first non-trivial positive integer solution (x_1, y_1) (first in the sense that x_1 or $x_1 + y_1\sqrt{d}$ is minimal) is called the *fundamental solution*, because it generates all the other solutions. In fact, if (x_1, y_1) is the fundamental solution of $x^2 - dy^2 = 1$, then the n^{th} positive solution (x_n, y_n) is defined by

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n \tag{2.1}$$

for any integer $n \geq 2$. (Furthermore, all non-trivial solutions can be obtained by considering the four cases $(\pm x_n, \pm y_n)$ for $n \geq 1$.)

There are several methods for finding the fundamental solution of Pell's equation $x^2 - dy^2 = 1$ for a positive non-square integer d. For example, the cyclic method known in India in the 12$^{\text{th}}$ century, and the slightly less efficient but more regular English method (17$^{\text{th}}$ century) produce all solutions of $x^2 - dy^2 = 1$ (see [3, pp. 30, 32]). But the most efficient method for finding the fundamental solution is based on the simple finite continued fraction expansion of $\sqrt{d}$. We can describe it as follows (see [1] and [4, p. 154]).

Let $[a_0; \overline{a_1, \ldots, a_r, 2a_0}\,]$ be the simple continued fraction expansion of $\sqrt{d}$ ($a_0 = \lfloor \sqrt{d} \rfloor$). Let $p_0 = a_0$, $p_1 = 1 + a_0 a_1$, $q_0 = 1$, $q_1 = a_1$, and

$$\begin{cases} p_n &= a_n p_{n-1} + p_{n-2}\,, \\ q_n &= a_n q_{n-1} + q_{n-2}\,, \end{cases} \qquad \text{for } n \geq 2.$$

If r is odd, the fundamental solution is $(x_1, y_1) = (p_r, q_r)$, where p_r/q_r is the r^{th} convergent of $\sqrt{d}$; if r is even, the fundamental solution is $(x_1, y_1) = (p_{2r+1}, q_{2r+1})$.

In connection with (1.1), it is well known ([6, Theorem 8.8, p. 146]) that if (u_1, v_1) is a solution of (1.1) and (x_1, y_1) is a solution of $x^2 - dy^2 = 1$, then (u, v) is a solution of (1.1), where

$$u + dv = (x_1 + dy_1)(u_1 + dv_1)\,. \tag{2.2}$$

However, in general ([6, p. 146, example below Theorem 8.8]), for $N \neq 1$ there is no a fundamental solution of $x^2 - dy^2 = \pm N$ (that is, a positive solution (u_1, v_1) such that for any positive solution (u, v), we have $(u, v) = (u_n, v_n)$ for some $n \in \mathbb{N}$). A general procedure to obtain the positive solutions of a solvable Pell equation $x^2 - dy^2 = N$ (for $N > 1$) can be found in [6, pp. 147–148, Theorem 8.9].

3 The Pell equation $x^2 - (k^2 - 2)y^2 = 2^t$.

First we consider the case $t = 0$; that is, the classical equation $x^2 - (k^2 - 2)y^2 = 1$.

Theorem 1 Let $d = k^2 - 2$ with $k \geq 2$.

(a) The continued fraction expansion of $\sqrt{d}$ is given by

$$\sqrt{d} = \begin{cases} [1; \overline{2}\,] & \text{if } k = 2\,, \\ [k-1; \overline{1, k-2, 1, 2k-2}\,] & \text{otherwise.} \end{cases} \tag{3.1}$$

(b) The fundamental solution of $x^2 - dy^2 = 1$ is

$$(x_1, y_1) = (k^2 - 1, k)\,. \tag{3.2}$$

Proof: (a) The case $k=2$ can be easily verified. Suppose that $k\geq 3$. Then (3.1) follows, because

$$\begin{aligned}\sqrt{k^2-2} &= k-1+\left(\sqrt{k^2-2}-(k-1)\right) = k-1+\cfrac{1}{\cfrac{1}{\sqrt{k^2-2}-(k-1)}}\\ &= k-1+\cfrac{1}{1+\cfrac{1}{\cfrac{\sqrt{k^2-2}+(k-2)}{2}}} = k-1+\cfrac{1}{1+\cfrac{1}{k-2+\cfrac{1}{\cfrac{2}{\sqrt{k^2-2}-(k-2)}}}}\\ &= k-1+\cfrac{1}{1+\cfrac{1}{k-2+\cfrac{1}{1+\cfrac{\sqrt{k^2-2}-(k-1)}{2k-3}}}}\\ &= k-1+\cfrac{1}{1+\cfrac{1}{k-2+\cfrac{1}{1+\cfrac{1}{\sqrt{k^2-2}+(k-1)}}}}\\ &= k-1+\cfrac{1}{1+\cfrac{1}{k-2+\cfrac{1}{1+\cfrac{1}{2k-2+\left(\sqrt{k^2-2}-(k-1)\right)}}}}.\end{aligned}$$

(b) The case $k=2$ of (3.2) follows because $(x,y)=(3,2)$ is clearly a minimum solution. On the other hand, using the method from part (a) above, for $k\geq 3$ we have $r=3$, $a_0=k-1$, $a_1=1$, $a_2=k-2$, $a_3=1$. Hence, $p_0=k-1$, $p_1=k$, $p_2=k^2-k-1$, $p_3=k^2-1$, $q_1=1$, $q_2=k-1$, and $q_3=k$. Thus, $(x_1,y_1)=(p_3,q_3)=(k^2-1,k)$. ■

Next we consider the general case.

Theorem 2 Let k, t, d be arbitrary integers with $k\geq 2$, $t\geq 0$, and $d=k^2-2$. Define a sequence $\{(u_n,v_n)\}$ of positive integers by

$$(u_1,v_1) = \begin{cases}\left(2^{(t-1)/2}k, 2^{(t-1)/2}\right) & \text{if } t \text{ is odd,}\\ \left(2^{t/2}(k^2-1), 2^{t/2}k\right) & \text{if } t \text{ is even.}\end{cases} \tag{3.3}$$

and, for $n\geq 2$,

$$(u_n,v_n) = (u_1x_{n-1}+dv_1y_{n-1}, v_1x_{n-1}+u_1y_{n-1}), \tag{3.4}$$

where $\{(x_n,y_n)\}$ is the sequence of positive solutions of $x^2-dy^2=1$. Then

(a) (u_n,v_n) is a solution of $x^2-dy^2=2^t$ for any integer $n\geq 1$.

(b) For $n\geq 1$, we have

$$\begin{cases}u_{n+1} = (k^2-1)u_n+(k^3-2k)v_n\,,\\ v_{n+1} = ku_n+(k^2-1)v_n\,.\end{cases} \tag{3.5}$$

(c) For $n \geq 1$, we have

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{cases} 2^{(t-1)/2} \begin{pmatrix} k^2-1 & k^3-2k \\ k & k^2-1 \end{pmatrix}^{n-1} \begin{pmatrix} k \\ 1 \end{pmatrix} & \text{if } t \text{ is odd,} \\ 2^{t/2} \begin{pmatrix} k^2-1 & k^3-2k \\ k & k^2-1 \end{pmatrix}^{n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } t \text{ is even.} \end{cases} \tag{3.6}$$

Proof: (a) Assume t is odd. We have that $(u_1, v_1) = \left(2^{(t-1)/2}k, 2^{(t-1)/2}\right)$ is a solution of $x^2 - dy^2 = 2^t$, because

$$\begin{aligned} u_1^2 - dv_1^2 &= \left(2^{(t-1)/2}k\right)^2 - (k^2-2)\left(2^{(t-1)/2}\right)^2 \\ &= 2^{t-1}k^2 - 2^{t-1}k^2 + 2 \cdot 2^{t-1} = 2^t . \end{aligned}$$

Similarly it can be shown that $(u_1, v_1) = \left(2^{t/2}(k^2-1), 2^{t/2}k\right)$ is a solution when t is even.

On the other hand, rewriting (3.4) as

$$u_n + v_n\sqrt{d} = (x_{n-1} + y_{n-1}\sqrt{d})(u_1 + v_1\sqrt{d}), \tag{3.7}$$

we see from (2.2) that (u_n, v_n) is also a solution for each $n \geq 2$. This can also be proved directly as follows:

$$\begin{aligned} u_n^2 - dv_n^2 &= (u_1x_{n-1} + dv_1y_{n-1})^2 - d(v_1x_{n-1} + u_1y_{n-1})^2 \\ &= u_1^2(x_{n-1}^2 - dy_{n-1}^2) - dv_1^2(x_{n-1}^2 - dy_{n-1}^2) \\ &= (x_{n-1}^2 - dy_{n-1}^2)(u_1^2 - dv_1^2) = 2^t . \end{aligned}$$

(b) Using repeatedly (2.1) and (3.7), we obtain

$$\begin{aligned} u_{n+1} + v_{n+1}\sqrt{d} &= (x_n + y_n\sqrt{d})(u_1 + v_1\sqrt{d}) \\ &= (x_1 + y_1\sqrt{d})^n(u_1 + v_1\sqrt{d}) \\ &= (x_1 + y_1\sqrt{d})\left[(x_1 + y_1\sqrt{d})^{n-1}(u_1 + v_1\sqrt{d})\right] \\ &= (x_1 + y_1\sqrt{d})\left[(x_{n-1} + y_{n-1}\sqrt{d})(u_1 + v_1\sqrt{d})\right] \\ &= (x_1 + y_1\sqrt{d})(u_n + v_n\sqrt{d}) , \end{aligned}$$

which is equivalent to (3.5).

(c) We can rewrite (3.5) in the form $\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} x_1 & dy_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix}$. Hence, proceeding by induction on $n \geq 1$, we obtain

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} x_1 & dy_1 \\ y_1 & x_1 \end{pmatrix}^{n-1} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} .$$

Now (3.6) follows from (3.3), because

$$\begin{pmatrix} k^2-1 \\ k \end{pmatrix} = \begin{pmatrix} x_1 & dy_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

■

Example 3.1 Let $k = 4$. Then the fundamental solution of $x^2 - 14y^2 = 1$ is $(x_1, y_1) = (15, 4)$, and some other solutions are

$$(x_2, y_2) = (449, 120), \quad (x_3, y_3) = (13455, 3596),$$
$$(x_4, y_4) = (403201, 107760), \quad (x_5, y_5) = (12082575, 3229204),$$
$$\text{and} \quad (x_6, y_6) = (362074049, 96768360).$$

Let $t = 6$. A solution of $x^2 - 14y^2 = 64$ is given by $(u_1, v_1) = (120, 32)$. Hence, using (3.5), we get

$$(u_2, v_2) = (3592, 960), \quad (u_3, v_3) = (107640, 28768),$$
$$(u_4, v_4) = (3225608, 862080), \quad (u_5, v_5) = (96660600, 25833632),$$
$$(u_6, v_6) = (2896592392, 774146880).$$

Problem. Prove or disprove that (u_1, v_1) is a fundamental solution of

$$x^2 - (k^2 - 2)y^2 = 2^t \quad (\text{for } t \geq 1).$$

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PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er avril 2008**. *Une étoile (⋆) après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

3263. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Les nombres de Fibonacci F_n et les nombres de Lucas L_n sont définis par les récurrences suivantes :

$$F_0 = 0, \quad F_1 = 1, \quad \text{et} \quad F_{n+1} = F_n + F_{n-1} \quad \text{pour } n \geq 1;$$
$$L_0 = 2, \quad L_1 = 1, \quad \text{et} \quad L_{n+1} = L_n + L_{n-1} \quad \text{pour } n \geq 1.$$

Soit n un entier positif. Montrer que

$$L_n L_{n+1} \leq 2 + \left(\sum_{k=1}^{n} L_k F_{2k}\right)^{\frac{1}{2}} \cdot \sum_{k=1}^{n} \frac{L_k^2}{\sqrt{F_k}}.$$

3264. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit M le point milieu de BC dans le triangle ABC et supposons que la bissectrice intérieure de l'angle BAC coupe BC en N. Montrer que $\angle BAC = 90^\circ + \angle MAN$ si et seulement si $b/c = 1 - 2\cos A$.

3265. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit $ABCD$ un trapèze de côtés parallèles AB et CD avec $AD = CD$ et $AC = BC$, AC et BD se coupant en E. Soit respectivement x, y et z les mesures des angles ABC, BDC et AED. Montrer que $y \leq 30^\circ$,

$$\tan y = \frac{2\tan x}{3 + \tan^2 x}, \quad \text{et} \quad \tan z = \frac{2\sin x + \sin 3x}{2\cos x + \cos 3x}.$$

3266. *Proposé par Michel Bataille, Rouen, France.*

Trouver tous les entiers positifs n ayant la propriété suivante : chaque fois que a et b sont des entiers tels que $ab + 1$ est un multiple de n, il en est de même pour $a + b$.

3267. *Proposé par Michel Bataille, Rouen, France.*

Dans un triangle non équilatéral ABC soit I le centre du cercle inscrit et O celui du cercle circonscrit. Désignons respectivement par X, Y et Z les points milieux de BC, CA et AB. Si $\pi(P)$ représente la projection d'un point P sur la droite OI, et si $\sigma_{MN}(P)$ représente la réflexion du point P par rapport à la droite MN, montrer que

$$\sigma_{YZ} \circ \pi(A) \;=\; \sigma_{ZX} \circ \pi(B) \;=\; \sigma_{XY} \circ \pi(C)\,.$$

3268. *Proposé par Bill Sands et John Wiest, Université de Calgary, Calgary, AB.*

Supposons donné une suite infini de cartes $C_1, C_2, \ldots$. Sur chaque carte est inscrite une série infinie de nombres réels non négatifs dont la somme vaut 1.

(a) Montrer qu'il existe un réarrangement D_1, D_2, ..., des cartes tel que la série $\sum\limits_{i=1}^{\infty} d_{ii}$ converge, où d_{ii} est le $i^{\text{ième}}$ nombre de la carte D_i.

(b)★ Est-ce qu'il existe un réarrangement tel que $\sum\limits_{i=1}^{\infty} d_{ii} \leq 1$?

[*Ed :* Comparer avec le problème 2620 [2002 : 127 ; 2005 : 319–326].]

3269. *Proposé par Pantelimon George Popescu, Bucarest, Roumanie et José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit n un entier positif. Montrer que

$$\exp\left(\frac{2^n}{n+1}\right) \sum_{k=1}^{n} \frac{k}{\exp\binom{n}{k}} \;\geq\; \binom{n+1}{2}\,.$$

3270. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit P un point quelconque à égale distance de deux droites k et ℓ. Soit A et B les projections orthogonales respectives de P sur k et ℓ. Montrer que pour tout $M \in k$ et $N \in \ell$, les énoncés suivants sont équivalents :

(i) $PN \perp BM$;

(ii) $PM \perp AN$;

(iii) $MN^2 = AM^2 + BN^2$.

3271. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit a, b et c nombres réels. Montrer que $|a+b|+|b+c|+|c+a| \leq 2$ si et seulement si $|a| \leq 1$, $|b| \leq 1$, $|c| \leq 1$, et $|a+b+c| \leq 1$.

3272. *Proposé par D.E. Prithwijit, University College Cork, République d'Irlande.*

Trouver tous les nombres naturels a et b tels que $a \mid (b^2+1)$ et $b \mid (a^2+1)$.

3273. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Sur les côtés du triangle ABC on dessine des triangles isocèles BMC, CNA et APB avec $MB = MC$, $NC = NA$ et $PA = PB$. Si $\angle BMC + \angle CNA + \angle APB = 360^\circ$, montrer que les angles du triangle MNP sont indépendants du triangle ABC.

3274. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit a, b et c trois nombres réels non négatifs. Montrer que

$$\frac{a^3}{2a^2+b^2}+\frac{b^3}{2b^2+c^2}+\frac{c^3}{2c^2+a^2} \geq \frac{a+b+c}{3}.$$

3275. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit x, y et z trois nombres réels non négatifs satisfaisant $x+y+z=3$, et soit $0 \le r \le 8$. Montrer que

$$\frac{1}{xy^2+r}+\frac{1}{yz^2+r}+\frac{1}{zx^2+r} \geq \frac{3}{1+r}.$$

. . . .

3263. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

The Fibonacci numbers F_n and Lucas numbers L_n are defined by the following recurrences:

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad \text{for } n \geq 1;$$
$$L_0 = 2, \quad L_1 = 1, \quad \text{and} \quad L_{n+1} = L_n + L_{n-1} \quad \text{for } n \geq 1.$$

Prove that for each positive integer n,

$$L_nL_{n+1} \leq 2+\left(\sum_{k=1}^{n} L_kF_{2k}\right)^{\frac{1}{2}} \cdot \sum_{k=1}^{n} \frac{L_k^2}{\sqrt{F_k}}.$$

3264. *Proposed by Virgil Nicula, Bucharest, Romania.*

Let M be the mid-point of BC in $\triangle ABC$, and let the interior angle bisector of $\angle BAC$ meet BC at N. Prove that $\angle BAC = 90^\circ + \angle MAN$ if and only if $b/c = 1 - 2\cos A$.

3265. *Proposed by Virgil Nicula, Bucharest, Romania.*

Let $ABCD$ be a trapezoid with $AB \parallel CD$ for which $AD = CD$ and $AC = BC$, and let E be the intersection of AC and BD. Let x, y, z denote the measures of angles ABC, BDC, AED, respectively. Show that $y \leq 30^\circ$,

$$\tan y \;=\; \frac{2\tan x}{3+\tan^2 x}\,, \quad \text{and} \quad \tan z \;=\; \frac{2\sin x + \sin 3x}{2\cos x + \cos 3x}\,.$$

3266. *Proposed by Michel Bataille, Rouen, France.*

Find all positive integers n with the following property: whenever a and b are integers such that $ab+1$ is a multiple of n, then $a+b$ is also a multiple of n.

3267. *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a non-equilateral triangle with circumcentre O and incentre I. Let X, Y, Z be the mid-points of BC, CA, AB, respectively. If $\pi(P)$ represents the projection of a point P onto the line OI, and $\sigma_{MN}(P)$ represents the reflection of the point P in the line MN, prove that

$$\sigma_{YZ}\circ\pi(A) \;=\; \sigma_{ZX}\circ\pi(B) \;=\; \sigma_{XY}\circ\pi(C)\,.$$

3268. *Proposed by Bill Sands and John Wiest, University of Calgary, Calgary, AB.*

You are given an infinite sequence of cards $C_1, C_2, \ldots$, on each of which is written an infinite series of non-negative real numbers which sums to 1.

(a) Prove that there is a reordering D_1, D_2, $\ldots$ of the cards such that the series $\sum\limits_{i=1}^{\infty} d_{ii}$ converges, where d_{ii} is the i^{th} term of the series on card D_i.

(b)★ Is there necessarily a reordering such that $\sum\limits_{i=1}^{\infty} d_{ii} \leq 1$?

[*Ed:* Compare with problem 2620 [2002 : 127; 2005 : 319–326].]

3269. *Proposed by Pantelimon George Popescu, Bucharest, Romania and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let n be a positive integer. Prove that

$$\exp\left(\frac{2^n}{n+1}\right)\sum_{k=1}^{n}\frac{k}{\exp\binom{n}{k}} \;\geq\; \binom{n+1}{2}\,.$$

3270. *Proposed by Virgil Nicula, Bucharest, Romania.*

Let k and ℓ be two straight lines, and let P be any point equidistant from them. Let A and B be the orthogonal projections of P onto k and ℓ, respectively. Prove that, for any $M \in k$ and $N \in \ell$, the following statements are equivalent:

(i) $PN \perp BM$;

(ii) $PM \perp AN$;

(iii) $MN^2 = AM^2 + BN^2$.

3271. *Proposed by Virgil Nicula, Bucharest, Romania.*

Let a, b, and c be real numbers. Prove that $|a+b|+|b+c|+|c+a| \leq 2$ if and only if $|a| \leq 1$, $|b| \leq 1$, $|c| \leq 1$, and $|a+b+c| \leq 1$.

3272. *Proposed by D.E. Prithwijit, University College Cork, Republic of Ireland.*

Characterize all natural numbers a and b such that $a \mid (b^2+1)$ and $b \mid (a^2+1)$.

3273. *Proposed by Virgil Nicula, Bucharest, Romania.*

On the sides of triangle ABC are mounted isosceles triangles BMC, CNA, and APB with $MB = MC$, $NC = NA$, and $PA = PB$. If $\angle BMC + \angle CNA + \angle APB = 360^\circ$, prove that the angles of $\triangle MNP$ are independent of $\triangle ABC$.

3274. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, and c be non-negative real numbers. Prove that

$$\frac{a^3}{2a^2+b^2}+\frac{b^3}{2b^2+c^2}+\frac{c^3}{2c^2+a^2} \;\geq\; \frac{a+b+c}{3}\,.$$

3275. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let x, y, and z be non-negative real numbers satisfying $x+y+z=3$, and let $0 \leq r \leq 8$. Prove that

$$\frac{1}{xy^2+r}+\frac{1}{yz^2+r}+\frac{1}{zx^2+r} \;\geq\; \frac{3}{1+r}\,.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3164. [2006 : 394, 396] *Proposed by Mihály Bencze, Brasov, Romania.*

Let P be any point in the plane of $\triangle ABC$. Let D, E, and F denote the mid-points of BC, CA, and AB, respectively. If G is the centroid of $\triangle ABC$, prove that

$$0 \leq 3PG+PA+PB+PC-2(PD+PE+PF) \leq \tfrac{1}{2}(AB+BC+CA)\,.$$

Composite of similar solutions by Michel Bataille, Rouen, France; and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

The left inequality has already been proven (see the solution of problem 3052 [2006 : 341]).

As in 3052, we set $\mathrm{a} = \overrightarrow{PA}$, $\mathrm{b} = \overrightarrow{PB}$, and $\mathrm{c} = \overrightarrow{PC}$. With this notation, the right inequality can be rewritten as

$$\begin{aligned} &|\mathrm{a}| + |\mathrm{b}| + |\mathrm{c}| + |\mathrm{a}+\mathrm{b}+\mathrm{c}| - |\mathrm{b}+\mathrm{c}| - |\mathrm{c}+\mathrm{a}| - |\mathrm{a}+\mathrm{b}| \\ &\qquad\qquad \leq \tfrac{1}{2}(|\mathrm{b}-\mathrm{a}| + |\mathrm{c}-\mathrm{b}| + |\mathrm{a}-\mathrm{c}|)\,, \end{aligned}$$

or

$$\begin{aligned} &|2\mathrm{a}| + |2\mathrm{b}| + |2\mathrm{c}| + |2(\mathrm{a}+\mathrm{b}+\mathrm{c})| \\ &\qquad \leq |\mathrm{b}-\mathrm{a}| + |\mathrm{c}-\mathrm{b}| + |\mathrm{a}-\mathrm{c}| + 2(|\mathrm{b}+\mathrm{c}| + |\mathrm{c}+\mathrm{a}| + |\mathrm{a}+\mathrm{b}|)\,. \end{aligned}$$

Now, the Triangle Inequality gives us

$$\begin{aligned} |2\mathrm{a}| &= |(\mathrm{a}+\mathrm{b}) - (\mathrm{b}-\mathrm{a})| \leq |\mathrm{a}+\mathrm{b}| + |\mathrm{b}-\mathrm{a}|\,, \\ |2\mathrm{b}| &= |(\mathrm{b}+\mathrm{c}) - (\mathrm{c}-\mathrm{b})| \leq |\mathrm{b}+\mathrm{c}| + |\mathrm{c}-\mathrm{b}|\,, \\ |2\mathrm{c}| &= |(\mathrm{c}+\mathrm{a}) - (\mathrm{a}-\mathrm{c})| \leq |\mathrm{c}+\mathrm{a}| + |\mathrm{a}-\mathrm{c}|\,, \\ |2(\mathrm{a}+\mathrm{b}+\mathrm{c})| &= |(\mathrm{a}+\mathrm{b}) + (\mathrm{b}+\mathrm{c}) + (\mathrm{c}+\mathrm{a})| \\ &\leq |\mathrm{a}+\mathrm{b}| + |\mathrm{b}+\mathrm{c}| + |\mathrm{c}+\mathrm{a}|\,. \end{aligned}$$

The result follows by adding the last four inequalities.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; and the proposer.

3165. [2006 : 394, 397] *Proposed by Mihály Bencze, Brasov, Romania.*

For any positive integer n, prove that there exists a polynomial $P(x)$, of degree at least $8n$, such that

$$\sum_{k=1}^{(2n+1)^2} |P(k)| \;<\; |P(0)|\,.$$

Essentially the same solution by Roy Barbara, Lebanese University, Fanar, Lebanon; Michel Bataille, Rouen, France; Richard I. Hess, Rancho Palos Verdes, CA, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and John Hawkins and David R. Stone, Georgia Southern University, Statesboro, GA, USA.

Let n be a positive integer. Let $P(x) = \prod_{k=1}^{(2n+1)^2} (x-k)$. Then $P(k) = 0$ for $k = 1, 2, \dots, (2n+1)^2$, and therefore,

$$\sum_{k=1}^{(2n+1)^2} |P(k)| \;=\; 0 \;<\; (2n+1)^2! \;=\; |P(0)|\,.$$

The degree of $P(x)$ is $(2n+1)^2 \geq 8n$ (note that this inequality is equivalent to $(2n-1)^2 \geq 0$).

Also solved by M.R. MODAK, Pune, India; and the proposer.

The solution by Modak was the same as the one above except that he defined $P(x)$ as the product of $x-k$ for $k = 2$ to $k = (2n+1)^2$ instead of $k = 1$ to $k = (2n+1)^2$. Thus, his polynomial $P(x)$ has degree $(2n+1)^2 - 1 = 4n(n+1)$. The proposer's solution was considerably more complicated, involving Chebyshev polynomials.

3166. [2004–118] *Proposed by Mihály Bencze and Marian Dinca, Brasov, Romania.*

Let P be an interior point of the triangle ABC. Denote by d_a, d_b, d_c the distances from P to the sides BC, CA, AB, respectively, and denote by D_A, D_B, D_C the distances from P to the vertices A, B, C, respectively. Further let P_A, P_B, and P_C denote the measures of $\angle BPC$, $\angle CPA$, and $\angle APB$, respectively.

Prove that

$$d_ad_b \sin\left(\tfrac{1}{2}(P_A+P_B)\right) + d_bd_c \sin\left(\tfrac{1}{2}(P_B+P_C)\right) + d_cd_a \sin\left(\tfrac{1}{2}(P_C+P_A)\right)$$
$$\leq\; \tfrac{1}{4}(D_BD_C \sin P_A + D_CD_A \sin P_B + D_AD_B \sin P_C)\,.$$

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let $[XYZ]$ represent the area of $\triangle XYZ$. Then the right side of the given inequality is simply $\frac{1}{2}[ABC]$.

Let the interior angle bisectors of $\angle BPC$, $\angle CPA$, and $\angle APB$ meet the sides BC, CA, and AB at A', B', and C', respectively. Then $PA' \geq d_a$, $PB' \geq d_b$, and $PC' \geq d_c$. Thus, the left side of the given inequality is less than twice the sum of $[A'PB']$, $[B'PC']$, and $[C'PA']$; that is, the left side of the given inequality is less than $2[A'B'C']$.

Therefore, it suffices to prove that $[A'B'C'] \leq \frac{1}{4}[ABC]$.

But $A'B'C'$ is a *Cevian triangle* of $\triangle ABC$; that is, AA', BB', and CC' are concurrent. This follows since

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} \;=\; \frac{PB}{PC} \cdot \frac{PC}{PA} \cdot \frac{PA}{PB} \;=\; 1\,.$$

The desired result then follows from the following theorem:

Theorem. Let AA', BB', and CC' be three concurrent Cevians of $\triangle ABC$. Then $[A'B'C'] \leq \frac{1}{4}[ABC]$.

Proof: Let

$$\lambda \;=\; \frac{AC'}{C'B}\,, \quad \mu \;=\; \frac{BA'}{A'C}\,, \quad \text{and} \quad \nu \;=\; \frac{CB'}{B'A}\,.$$

From Ceva's Theorem, we have $\lambda\mu\nu = 1$. Then

$$[BA'C'] \;=\; \frac{\mu[ABC]}{(1+\mu)(1+\lambda)}\,, \quad [CB'A'] \;=\; \frac{\nu[ABC]}{(1+\nu)(1+\mu)}\,,$$
$$\text{and} \quad [AC'B'] \;=\; \frac{\lambda[ABC]}{(1+\lambda)(1+\nu)}\,.$$

Hence,

$$\begin{aligned}
\frac{[A'B'C']}{[ABC]} &= 1 - \frac{\mu}{(1+\mu)(1+\lambda)} - \frac{\nu}{(1+\nu)(1+\mu)} - \frac{\lambda}{(1+\lambda)(1+\nu)} \\
&= 1 - \frac{\mu(1+\nu)+\nu(1+\lambda)+\lambda(1+\mu)}{(1+\lambda)(1+\mu)(1+\nu)} \\
&= 1 - \frac{\lambda+\mu+\nu+\lambda\mu+\mu\nu+\nu\lambda}{(1+\lambda)(1+\mu)(1+\nu)} \\
&= \frac{1+\lambda\mu\nu}{1+\lambda+\mu+\nu+\lambda\mu+\mu\nu+\nu\lambda+\lambda\mu\nu} \\
&= \frac{2}{2+(\lambda+1/\lambda)+(\mu+1/\mu)+(\nu+1/\nu)} \qquad \text{since } \lambda\mu\nu = 1
\end{aligned}$$

which is obviously less than or equal to $\frac{2}{8} = \frac{1}{4}$ because each of the bracketed expressions is at least 2. ■

Also solved by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

3167. [2006 : 395, 397] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let ABC be a non-obtuse triangle with circumradius R. If a, b, c are the lengths of the sides opposite angles A, B, C, respectively, prove that

$$a\cos^3 A + b\cos^3 B + c\cos^3 C \;\le\; \frac{abc}{4R^2}\,.$$

Composite of similar solutions by Mohammed Aassila, Strasbourg, France; and Vedula N. Murty, Dover, PA, USA.

Let S be the area of triangle ABC. Since

$$S \;=\; \frac{abc}{4R} \;=\; \frac{1}{2}R^2\sum_{\text{cyclic}}\sin 2A\,,$$

$$\frac{a}{\sin A} \;=\; \frac{b}{\sin B} \;=\; \frac{c}{\sin C} \;=\; 2R\,,$$

$$\text{and} \quad \sum_{\text{cyclic}}\sin 4A \;=\; -4\sin 2A\sin 2B\sin 2C\,,$$

we have

$$\begin{aligned}
\sum_{\text{cyclic}} a\cos^3 A &= \sum_{\text{cyclic}}(2R\sin A)\cos^3 A \;=\; R\sum_{\text{cyclic}}\sin 2A\cos^2 A \\
&= \frac{1}{2}R\sum_{\text{cyclic}}\sin 2A(1+\cos 2A) \\
&= \frac{1}{2}R\sum_{\text{cyclic}}\sin 2A + \frac{1}{4}R\sum_{\text{cyclic}}\sin 4A \\
&= \frac{abc}{4R^2} - R\sin 2A\sin 2B\sin 2C\,.
\end{aligned}$$

Now we note that $\sin 2A\sin 2B\sin 2C \ge 0$ because $\triangle ABC$ is non-obtuse. Thus, we obtain the desired inequality.

Equality holds if and only if the triangle is right-angled.

Also solved by MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Howard observed that

$$\sum_{\text{cyclic}} a\cos^3 A \;>\; \frac{abc}{4R^2}$$

if and only if the triangle is obtuse, a fact that follows easily from the featured solution as well.

3168. [2006 : 395, 397] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $x_1, x_2, \ldots, x_n$ be positive real numbers satisfying $\prod_{i=1}^{n} x_i = 1$. Prove that

$$\sum_{i=1}^{n} x_i^n(1+x_i) \geq \frac{n}{2^{n-1}} \prod_{i=1}^{n}(1+x_i)\,.$$

Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

By the Power–Mean Inequality, we have $\sqrt[n]{\frac{1}{2}(a^n+1^n)} \geq \frac{1}{2}(a+1)$; this is equivalent to $a^n+1 \geq \frac{1}{2^{n-1}}(a+1)^n$. Using this result as well as the AM–GM Inequality and the given condition $x_1x_2\cdots x_n = 1$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n} x_i^n(1+x_i) &= x_1^n+\cdots+x_n^n+x_1^{n+1}+\cdots+x_n^{n+1} \\
&\geq x_1^n+\cdots+x_n^n+n\sqrt[n]{(x_1x_2\cdots x_n)^{n+1}} \\
&= x_1^n+\cdots+x_n^n+n = (x_1^n+1)+\cdots+(x_n^n+1) \\
&\geq \frac{1}{2^{n-1}}\left((x_1+1)^n+\cdots+(x_n+1)^n\right) \\
&\geq \frac{n}{2^{n-1}}\sqrt[n]{(x_1+1)^n(x_2+1)^n\cdots(x_n+1)^n} \\
&= \frac{n}{2^{n-1}}\prod_{i=1}^{n}(1+x_i)\,.
\end{aligned}$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Also solved by MOHAMMED AASSILA, Strasbourg, France; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Cao Minh remarked that the case $n = 3$ is problem 11 (Russia) of IMO Short List 1998.

3169. [2006 : 395, 397] *Proposed by Vesselin Dimitrov, National High-school of Mathematics and Science, Sofia, Bulgaria.*

Let A be a finite set of real numbers such that each $a \in A$ is uniquely expressible as $a = b + c$, where $b, c \in A$ and $b \leq c$.

(a) Prove that there exist distinct elements $a_1, a_2, \ldots, a_k \in A$ such that $a_1 + a_2 + \cdots + a_k = 0$.

(b)★ Does this necessarily hold if it is no longer assumed that each representation $a = b + c$ is unique?

No correct solutions were received for either part (a) or (b), so this problem remains open.

The proposer remarked that there are many finite sets $A \subset \mathbb{Z}$ for which the given condition holds. He claims, for example, that for each $n \in \mathbb{N}$, the set $\{-2^{n+1} + 2^k + 1,\ 2^k \mid k = 0, 1, 2, \ldots, n\}$ satisfies the requirement. [*Ed.*: Note that

$$\begin{aligned} 2^k &= 2^{k-1} + 2^{k-1} \qquad \text{if } k \geq 1\,, \\ 2^0 = 1 &= \left(-2^{n+1} + 2^n + 1\right) + 2^n\,, \\ -2^{n+1} + 2^k + 1 &= \left(-2^{n+1} + 2^{k-1} + 1\right) + 2^{k-1} \quad \text{if } k \geq 1\,, \\ -2^{n+1} + 2^0 + 1 &= -2^{n+1} + 2 = 2\left(-2^{n+1} + 2^n + 1\right). \end{aligned}$$

It is not difficult to verify that each of these representations is unique.]

3170. [2006 : 395, 398] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a and b be real numbers satisfying $0 \leq a \leq \frac{1}{2} \leq b \leq 1$. Prove that

(a) $2(b - a) \leq \cos \pi a - \cos \pi b$;

(b) $(1 - 2a) \cos \pi b \leq (1 - 2b) \cos \pi a$.

Solution by Michel Bataille, Rouen, France.

(a) Let $f(x) = 2x + \cos \pi x$. The proposed inequality can then be expressed as $f(b) \leq f(a)$.

We have $f'(x) = 2 - \pi \sin \pi x$ and $f''(x) = -\pi^2 \cos \pi x$. Hence, f' is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$. Since $f'(0) = f'(1) = 2$ and $f'\left(\frac{1}{2}\right) = 2 - \pi < 0$, there exist α and β with $0 < \alpha < \frac{1}{2} < \beta < 1$, such that $f'(\alpha) = f'(\beta) = 0$ and $f'(x) < 0$ if and only if $x \in (\alpha, \beta)$.

Thus, f is increasing on $[0, \alpha]$ and $[\beta, 1]$, and decreasing on $[\alpha, \beta]$. Since $f(0) = f\left(\frac{1}{2}\right) = f(1) = 1$, we see that $f(x) \geq 1$ for $x \in \left[0, \frac{1}{2}\right]$ and $f(x) \leq 1$ for $x \in \left[\frac{1}{2}, 1\right]$. In particular, $f(b) \leq 1 \leq f(a)$, and the result follows.

(b) (Modified slightly by the editor). The proposed inequality is false; for example, if $a = \frac{1}{4}$ and $b = 1$, then the inequality would imply that $-\frac{1}{2} \leq -\frac{\sqrt{2}}{2}$, which is absurd.

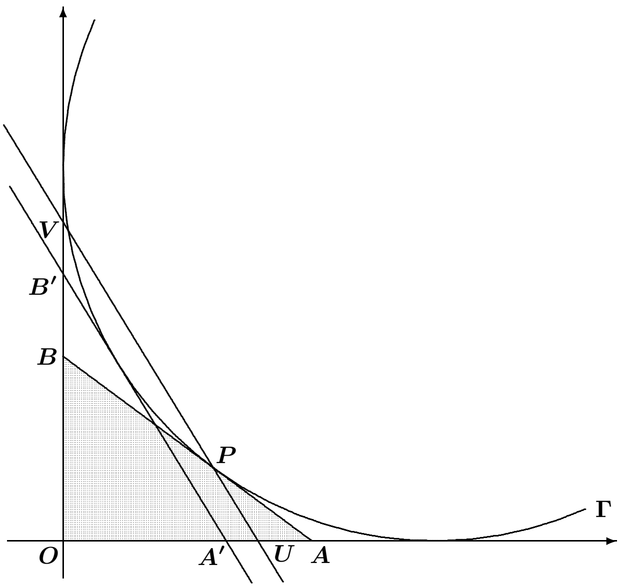
Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer (part (a) *only).*

All the solvers noticed that the inequality in (b) *is incorrect. Curtis commented that the inequality does hold sometimes (for example, when $a = 0$ and $b = \frac{3}{4}$); thus, one cannot simply reverse it.*

3171. [2006 : 395, 398] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given a point P in the first quadrant, it is known that the line segment in the first quadrant joining the coordinate axes, passing through P, and having minimum length (Philo's line) is not constructible using straightedge and compass. However, the line which (together with the two axes) defines a triangle in the first quadrant with minimum perimeter is constructible. Give such a construction.

I. *Solution by Claudio Arconcher, Jundiaí, Brazil.*



Claim. The hypotenuse of the triangle of minimum perimeter is the tangent at P to the circle through P, call it Γ, that is tangent to the positive x- and y-axes and separated by that tangent from the origin O.

Proof: Let the line through P tangent to Γ meet the x-axis at A and the y-axis at B. Let ℓ be any other line through P intersecting the positive axes at points U and V, say. Then the line parallel to UV and tangent to Γ intersects the axes at A' and B' with $OA' < OU$ and $OB' < OV$. Since $OA + OB + AB = OA' + OB' + A'B'$, which equals twice the length of the tangents to Γ from O, we have

$$OA + OB + AB \;=\; OA' + OB' + A'B' \;<\; OU + OV + UV\,,$$

as claimed. ∎

Construction. First construct an arbitrary circle Γ' that is tangent to the positive x- and y-axes (whose centre C' is an arbitrary point of the line $y = x$), and call P' the point closest to O where OP intersects Γ'. Define C to be the point where the line parallel to $P'C'$ through P intersects OC'. Then C is the centre of Γ (because the dilatation with centre O that takes C' to C will take P' to P, and take Γ' and its points of tangency with the axes to Γ and its tangency points on the axes).

II. *Composite of similar solutions by Peter Y. Woo, Biola University, La Mirada, CA, USA; and the proposer.*

Analysis. Let the axes meet at O, and let the line segment through $P(a,b)$ meet the x-axis at A and the y-axis at B. Define $\theta = \angle BAO$. Without loss of generality assume that $a \geq b$. Then the perimeter of triangle OAB is

$$p(\theta) \;=\; a(1+\sec\theta+\tan\theta)+b(1+\csc\theta+\cot\theta)\,.$$

Its derivative satisfies

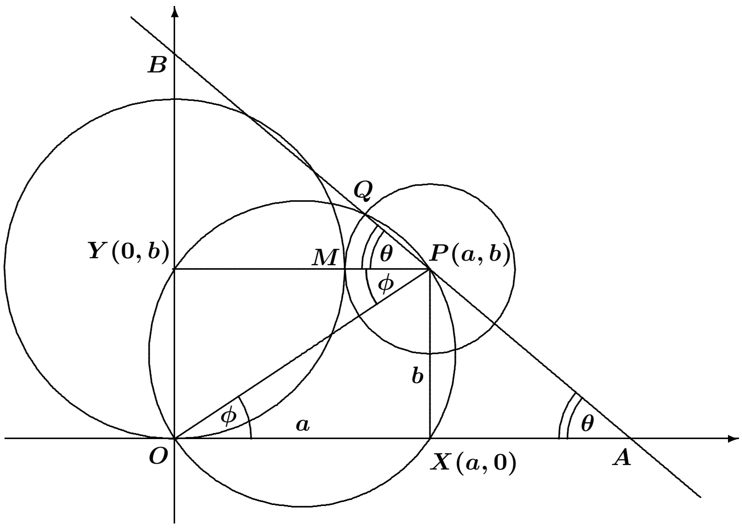
$$p'(\theta) \;=\; \frac{a}{1-\sin\theta}-\frac{b}{1-\cos\theta}\,.$$

The geometry indicates that the minimum perimeter occurs when the derivative is zero, which means $a\cos\theta - b\sin\theta = a - b$, or

$$\frac{a\cos\theta - b\sin\theta}{\sqrt{a^2+b^2}} \;=\; \frac{a-b}{\sqrt{a^2+b^2}}\,.$$

With $\phi = \angle AOP$, so that $\cos\phi = \dfrac{a}{\sqrt{a^2+b^2}}$ and $\sin\phi = \dfrac{b}{\sqrt{a^2+b^2}}$, the last equation can be interpreted as

$$\cos(\phi+\theta) \;=\; \frac{a-b}{\sqrt{a^2+b^2}}\,.$$



Construction. Construct the circle with diameter OP, cutting the y-axis again at $Y(0,b)$. Construct the circle with centre Y and radius YO, cutting the segment YP at M. Draw the circle with centre P and radius PM, cutting the first circle at Q (between P and B). Then PQ is the desired line that hits the axes at A and B and determines the triangle OAB of minimum perimeter. (Since $PQ = a - b$ and $PQ \perp OQ$, we get $\cos\angle QPO = \frac{a-b}{\sqrt{a^2+b^2}}$; whence, the line PQ makes an angle of θ with the x-axis such that $\theta + \phi = \angle OPQ$.)

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; and CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA.

3172. [2006 : 396, 398] *Proposed by Vincenţiu Rădulescu, University of Craiova, Craiova, Romania.*

Let f be a positive continuous function defined on $(0,\infty)$ such that $\liminf\limits_{x\to\infty} f(x) > 0$. Prove that there is no positive, twice differentiable function g defined on $[0,\infty)$ which satisfies $g'' + f\circ g = 0$.

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA, modified by the editor.

Suppose that such a function g exists. Then $f\circ g$ is continuous, and therefore g'' is also continuous. Since f is positive on $(0,\infty)$, the function g'' is negative on $(0,\infty)$, and g' is decreasing.

Now suppose that $g'(\alpha) = -k < 0$ for some $\alpha > 0$. Then $g'(x) \le -k$ for all $x \ge \alpha$. Therefore, $g(x) \le g(\alpha) - k(x-\alpha)$ for all $x \ge \alpha$, implying that g is eventually negative, a contradiction. Hence, $g'(x) \ge 0$ on $[0,\infty)$ and g is increasing.

Since g is positive and increasing, $\lim\limits_{x\to\infty} g(x) = \infty$ or $\lim\limits_{x\to\infty} g(x) = M$ for some positive real number M. If $\lim\limits_{x\to\infty} g(x) = \infty$, then

$$\liminf_{x\to\infty} f\big(g(x)\big) \;=\; \liminf_{y\to\infty} f(y) \;>\; 0\,.$$

If $\lim\limits_{x\to\infty} g(x) = M$, then, using the continuity of f, we have

$$\liminf_{x\to\infty} f\big(g(x)\big) \;=\; \lim_{y\to M} f(y) \;=\; f(M) \;>\; 0\,.$$

Thus, in both cases, $\liminf\limits_{x\to\infty} f\big(g(x)\big) > 0$.

Now, since $g'' = -f\circ g$, we have

$$\limsup_{x\to\infty} g''(x) \;=\; -\liminf_{x\to\infty} f\big(g(x)\big) \;<\; 0\,.$$

Therefore, there exist $\delta > 0$ and $\beta > 0$ such that $g''(x) \le -\delta$ for all $x \ge \beta$. Then $g'(x) \le g'(\beta) - \delta(x-\beta)$ for all $x \ge \beta$, implying that g' is eventually negative, a contradiction.

Hence, such a function g does not exist.

Also solved by APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; KEE-WAI LAU, Hong Kong, China; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

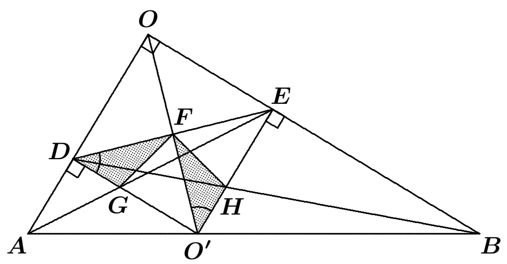
WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria, commented that this problem is Aufgabe 1224 of the Swiss journal Elemente der Matematik, *and that a solution can be found in the "Aufgaben" section of issue No. 4 of Vol. 61 (2006).*

3173. [2006 : 396, 398] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let OAB be a right triangle with right angle at O. Let OO' be the bisector of angle O, with O' on AB. Let D and E be the feet of the perpendiculars from O' to the legs OA and OB, respectively. Let $F = OO' \cap DE$, $G = AE \cap O'D$, and $H = BD \cap O'E$.

Prove that $\triangle FGH$ is an isosceles right triangle with right angle at F.

Composite of similar solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.



Since O' lies on the internal bisector of $\angle AOB$, we have $DO' = O'E$, and therefore, the rectangle $ODO'E$ is a square. Hence, $FD = FO'$ and $\angle FDG = \angle FO'H = 45°$. Since $\triangle DAG$ and $\triangle OAE$ are similar, as are $\triangle HO'B$ and $\triangle DAB$, we obtain

$$\frac{DG}{AD} = \frac{OE}{AO} = \frac{OD}{AO} = \frac{O'B}{AB} = \frac{O'H}{AD}\,.$$

Hence, $DG = O'H$. It follows that $\triangle FDG$ and $\triangle FO'H$ are congruent. Thus, $FG = FH$ and $\angle DFG = \angle O'FH$, which implies that $\angle GFH = 90°$ (because $\angle DFO' = 90°$). Consequently, $\triangle FGH$ is isosceles with a right angle at F.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3174. [2006 : 396, 398] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Given $\triangle ABC$, we define A' to be the point where the internal angle bisector of angle A meets the side BC. Let B' and C' be the feet of the perpendiculars from A' to the sides AC and AB, respectively. Prove that BB' and CC' intersect on the altitude from A.

Composite of similar solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

We will use directed distances in Ceva's Theorem to avoid any need for special cases. From the congruent right triangles $AC'A'$ and $AB'A'$, we deduce that $AC' = AB'$; that is,

$$\frac{AC'}{B'A} = 1\,.$$

Let D be the foot of the altitude from A. From the similar right triangles $C'BA'$ and DBA, we have

$$\frac{BD}{C'B} = \frac{AB}{BA'}\,,$$

and from the similar right triangles $A'B'C$ and ADC, we have

$$\frac{CB'}{DC} = \frac{A'C}{AC}\,.$$

Multiplying together these three equations, we obtain

$$\frac{AC'}{B'A}\cdot\frac{BD}{C'B}\cdot\frac{CB'}{DC} = \frac{AB}{AC}\cdot\frac{A'C}{BA'}\,.$$

Since A' lies on the bisector of angle A, we see that A' divides the segment BC in the ratio $AB : AC$; whence, $\frac{BA'}{A'C} = \frac{AB}{AC}$; that is,

$$\frac{AB}{AC}\cdot\frac{A'C}{BA'} = 1\,.$$

We conclude that

$$\frac{AC'}{C'B}\cdot\frac{BD}{DC}\cdot\frac{CB'}{B'A} = 1\,,$$

and the desired result follows from the converse of Ceva's Theorem. [Almost! See the remarks following the list of solvers.]

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; MICHAEL PARMENTER, Memorial University of Newfoundland, *St. John's, NL; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.*

Most solvers used some variant of the featured solution, but only Bataille and Parmenter noted that the converse of Ceva's Theorem asserts that the lines AD, BB', and CC' are **parallel or concurrent**. *Here is how Bataille completed the argument: $\angle A'B'C = 90°$; hence, the foot B'' of the perpendicular from B' to BC lies between A' and C; since A' lies between B and C, it follows that $B'' \neq B$, and the line $B'B$ is not parallel to AD. Consequently, AD, BB', CC' must be concurrent.*

3175. [2006 : 396, 398] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let $\triangle ABC$ be a triangle with $\angle B > 90°$ and $\angle A < 60°$. Let P be a point on the side AB such that $\angle CPB = 60°$. Let D be the point on CP which also lies on the interior angle bisector of $\angle A$. If $\angle CBD = 30°$, prove that CP is a trisector of angle ACB.

I. *Solution by Apostolis K. Demis, Varvakeio High School, Athens, Greece.*

Let AY be the bisector of $\angle CAB$, let $PM \perp AB$ with M on AC, let CN be the bisector of $\angle ACP$, let F be the point on AM with $\angle FPA = 60°$, let H be the point of intersection of CN and PM, and let E be the point of intersection of CN and AD. Denote the angles of $\triangle ABC$ by α, β, and γ as usual.

It is clear that $\angle FPM = \angle MPC = 30°$. From $\triangle PAC$, we obtain $\alpha + \angle ACP = 60°$; then $\angle ACN = \angle NCP = \frac{1}{2}\angle ACP = 30° - \frac{1}{2}\alpha$. From $\triangle APD$, we get $\frac{1}{2}\alpha + \angle ADP = 60°$; then $\angle ADP = 60° - \frac{1}{2}\alpha$.

In $\triangle FPC$, the line PM is the bisector of $\angle FPC$ and the line CN is the bisector of $\angle ACP$. Thus, H is the incentre of $\triangle FPC$. Hence, the line FH is the bisector of $\angle MFP$. From $\triangle APF$, we see that $\angle PFM = \alpha + 60°$, so that $\angle MFH = \angle HFP = \frac{1}{2}\alpha + 30°$.

From $\triangle FHC$, we obtain

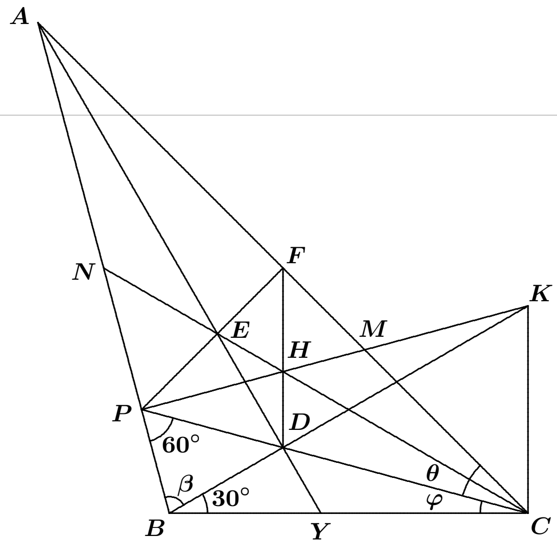
$$\begin{aligned} \angle FHE &= \angle FCH + \angle CFH \\ &= \left(30° - \tfrac{1}{2}\alpha\right) + \left(30° + \tfrac{1}{2}\alpha\right) \;=\; 60°\,. \end{aligned}$$

Thus, in $\triangle FHP$, we have

$$\begin{aligned} \angle EHP &= 180° - \angle HPF - \angle HFP - \angle EHF \\ &= 180° - 30° - \left(\tfrac{1}{2}\alpha + 30°\right) - 60° \;=\; 60° - \tfrac{1}{2}\alpha\,. \end{aligned}$$

Therefore, $\angle EDP = \angle ADP = 60° - \frac{1}{2}\alpha = \angle EHP$, which implies that quadrilateral $EHDP$ is cyclic. Thus, $\angle CHD = \angle EPD = 60°$, and hence, the points F, H, and D are collinear.

Let PM intersect BD at K. If $\angle DBC = 30°$, then quadrilateral $PBCK$ is cyclic, since $\angle KPC = \angle KBC = 30°$. Thus, $\angle DCB = \angle PKB$ and $\angle CKB = \angle CPB = 60°$. From above, we have $\angle CHD = 60°$; whence, $\angle CKB = \angle CHD$. Therefore, quadrilateral $KHDC$ is cyclic, which implies that $\angle HKD = \angle HCD$. It follows that $\angle DCB = \angle HCD = \angle ACH$.



II. *Solution by Geoffrey A. Kandall, Hamden, CT, USA.*

Set $\theta = \angle ACD$, $\varphi = \angle DCB$, and $\beta = \angle PBD$. From $\triangle PBC$, we see that $\beta + \varphi = 90^\circ$.

From the trigonometric form of Ceva's Theorem, we have

$$\frac{\sin \angle BAD}{\sin \angle DAC} \cdot \frac{\sin \theta}{\sin \varphi} \cdot \frac{\sin 30^\circ}{\sin \beta} \;=\; 1\,.$$

Hence,

$$\sin \theta \;=\; 2 \sin \varphi \sin \beta \;=\; 2 \sin \varphi \cos \varphi \;=\; \sin 2\varphi\,.$$

Since the angles θ and 2φ are not supplementary ($\theta < 60^\circ$ and $\varphi < 30^\circ$), we conclude that $\theta = 2\varphi$.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; M.R. MODAK, Pune, India; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3177. [2006 : 462, 464] *Proposed by Mihály Bencze and Marian Dinca, Brasov, Romania.*

Let P be any interior point of triangle $A_1A_2A_3$. Let T_1, T_2, T_3 denote the projections of P onto the sides A_2A_3, A_3A_1, A_1A_2, respectively, and let H_1, H_2, H_3 denote the orthocentres of triangles $A_1T_2T_3$, $A_2T_3T_1$, $A_3T_1T_2$, respectively. Prove that the lines H_1T_1, H_2T_2, H_3T_3 are concurrent.

A composite of similar solutions by Apostolis K. Demis, Varvakeio High School, Athens, Greece; and Taichi Maekawa, Takatsuki City, Osaka, Japan.

Because the lines T_2H_1 and PT_3 are both perpendicular to A_1A_2, they are parallel. Likewise, $T_3H_1 \parallel PT_2$; whence $H_1T_2PT_3$ is a parallelogram. In the same way, $H_2T_3PT_1$ is a parallelogram. Consequently, H_1T_2 is parallel and equal to its opposite side T_3P, which is parallel and equal to its opposite side H_2T_1. It follows that $H_1H_2T_1T_2$ is a parallelogram, so that

diagonals H_1T_1 and H_2T_2 have a common mid-point.

Similarly, using parallelograms $H_2T_3PT_1$ and $H_3T_1PT_2$, we deduce that $H_2H_3T_2T_3$ is a parallelogram; whence

diagonals H_2T_2 and H_3T_3 have a common mid-point.

Consequently, the segments H_1T_1, H_2T_2, and H_3T_3 have a common mid-point—the lines they determine are concurrent, as desired.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain (2 solutions); MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

In their second solution Bellot Rosado and López Chamorro determine that, with $A_1A_2A_3$ as the triangle of reference, if P has trilinear coordinates (p, q, r), then the common point of H_1T_1, H_2T_2, and H_3T_3 has coordinates

$$(p + r\cos B + q\cos C,\ q + p\cos C + r\cos A,\ r + q\cos A + p\cos B)\,.$$

If instead you let (p, q, r) be the areal coordinates of P, then Bataille shows that the common point has areal coordinates

$$(1-p, 1-q, 1-r)\,,$$

from which he deduces that the centroid of $\triangle T_1T_2T_3$ lies two-thirds of the way from P to the common point.